

A PROOF OF THE SHUFFLE CONJECTURE

ERIK CARLSSON AND ANTON MELLIT

ABSTRACT. We present a proof of the compositional shuffle conjecture [HMZ12], which generalizes the famous shuffle conjecture for the character of the diagonal coinvariant algebra [HHLRU05]. We first formulate the combinatorial side of the conjecture in terms of certain operators on a graded vector space V_* whose degree zero part is the ring of symmetric functions $\text{Sym}[X]$ over $\mathbb{Q}(q, t)$. We then extend these operators to an action of an algebra $\tilde{\mathbb{A}}$ acting on this space, and interpret the right generalization of the ∇ using an involution of the algebra which is antilinear with respect to the conjugation $(q, t) \mapsto (q^{-1}, t^{-1})$.

1. INTRODUCTION

The shuffle conjecture of Haglund, Haiman, Loehr, Remmel, and Ulyanov [HHLRU05] predicts a combinatorial formula for the Frobenius character $\mathcal{F}_{R_n}(X; q, t)$ of the diagonal coinvariant algebra R_n in n pairs of variables, which is a symmetric function in infinitely many variables with coefficients in $\mathbb{Z}_{\geq 0}[q, t]$. By a result of Haiman [Hai02], the Frobenius character is given explicitly by

$$\mathcal{F}_{R_n}(X; q, t) = (-1)^n \nabla e_n[X],$$

where up to a sign convention, ∇ is the operator which is diagonal in the modified Macdonald basis defined in [BGHT99]. The original shuffle conjecture states

$$(1) \quad (-1)^n \nabla e_n[X] = \sum_{\pi} \sum_{w \in \mathcal{WP}_{\pi}} t^{\text{area}(\pi)} q^{\text{dinv}(\pi, w)} x_w.$$

Here π is a Dyck path of length n , and w is some extra data called a “word parking function” depending on π . The functions $(\text{area}, \text{dinv})$ are statistics associated to a Dyck path and a parking function, and x_w is a monomial in the variables x . They proved that this sum, denoted $D_n(X; q, t)$, is symmetric in the x variables and so does at least define a symmetric function. They furthermore showed that it included many previous conjectures and results about the q, t -Catalan numbers, and other special cases [GM96; GH02; Hag03; EHKK03; Hag04]. Remarkably, $D_n(X; q, t)$ had not even been proven to be symmetric in the q, t variables until now, even though the symmetry of $\mathcal{F}_{R_n}(X; q, t)$ is obvious. For a thorough introduction to this topic, see Haglund’s book [Hag08].

In [HMZ12] Haglund, Morse, and Zabrocki conjectured a refinement of the original conjecture which partitions $D_n(X; q, t)$ by specifying the points where the Dyck path touches the diagonal called the “compositional shuffle conjecture.” The refined

conjecture states

$$(2) \quad \nabla (C_\alpha[X; q]) = \sum_{\text{touch}(\pi)=\alpha} \sum_{w \in \mathcal{WP}_\pi} t^{\text{area}(\pi)} q^{\text{dinv}(\pi, w)} x_w.$$

Here α is a composition, i.e. a finite list of positive integers specifying the gaps between the touch points of π . The function $C_\alpha[X; q]$ is defined below as a composition of creation operators for Hall-Littlewood polynomials in the variable $1/q$. They proved that

$$\sum_{|\alpha|=n} C_\alpha[X; q] = (-1)^n e_n[X],$$

implying that (2) does indeed generalize (1). The right hand side of (2) will be denoted by $D_\alpha(X; q, t)$. A desirable approach to proving (2) would be to determine a recursive formula for $D_\alpha(X; q, t)$, and interpret the result in terms of some commutation relations for ∇ . Indeed, this approach has been applied in some important special cases, see [GH02; Hic12]. Unfortunately, no such recursion is known in the general case, and so an even more refined function is needed.

In this paper, we will construct the desired refinement as an element of a larger vector space V_k of symmetric functions over $\mathbb{Q}(q, t)$ with k additional variables y_i adjoined, where k is the length of the composition α ,

$$N_\alpha \in V_k = \text{Sym}[X][y_1, \dots, y_k].$$

In our first result, (Theorem 4.6), we will explain how to recover $D_\alpha(X; q, t)$ from N_α . and prove that N_α satisfies a recursion that completely determines it. We then define a pair of algebras \mathbb{A} and \mathbb{A}^* which are isomorphic by an antilinear isomorphism with respect to the conjugation $(q, t) \rightarrow (q^{-1}, t^{-1})$, as well as an explicit action of each on the direct sum $V_* = \bigoplus_{k \geq 0} V_k$. We will then prove that there is an antilinear involution N on V_* which intertwines the two actions (Theorem 7.2), and represents an involutive automorphism on a larger algebra $\mathbb{A}, \mathbb{A}^* \subset \tilde{\mathbb{A}}$. This turns out to be the essential fact that relates the N_α to ∇ .

The compositional shuffle conjecture (Theorem 7.3), then follows as a simple corollary from the following properties:

- (i) There is a surjection coming from \mathbb{A}, \mathbb{A}^*

$$d_-^k : V_k \rightarrow V_0 = \text{Sym}[X]$$

which maps a monomial y_α in the y variables to an element $B_\alpha[X; q]$ which is similar to $C_\alpha[X; q]$, and maps N_α to $D_\alpha(X; q, t)$, up to a sign.

- (ii) The involution N commutes with d_- , and maps y_α to N_α .
- (iii) The restriction of N to $V_0 = \text{Sym}[X]$ agrees with ∇ composed with a conjugation map which essentially exchanges the $B_\alpha[X; q]$ and $C_\alpha[X; q]$.

It then becomes clear that these properties imply (2).

While the compositional shuffle conjecture is clearly our main application, the shuffle conjecture has been further generalized in several remarkable directions such as the rational compositional shuffle conjecture, and relationships to knot invariants, double affine Hecke algebras, and the cohomology of the affine Springer fibers,

see [BGLX14; GORS14; GN15; Neg13; Hik14; SV11; SV13]. We hope that future applications to these fascinating topics will be forthcoming.

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2. THE COMPOSITIONAL SHUFFLE CONJECTURE

2.1. Plethystic operators. A λ -ring is a ring R with a family of ring endomorphisms $(p_i)_{i \in \mathbb{Z}_{>0}}$ satisfying

$$p_1[x] = x, \quad p_m[p_n[x]] = p_{mn}[x], \quad (x \in R, \quad m, n \in \mathbb{Z}_{>0}).$$

Unless stated otherwise the endomorphisms are defined by $p_n(x) = x^n$ for each generator x , and every variable in this paper is considered a generator. The ring of symmetric functions over the λ -ring $\mathbb{Q}(q, t)$ is a free λ -ring with generator $X = x_1 + x_2 + \cdots$, and will be denoted $\text{Sym}[X]$. We will employ the standard notation used for plethystic substitution defined as follows: given an element $F \in \text{Sym}[X]$ and A in some λ -ring R , the plethystic substitution $F[A]$ is the image of the homomorphism from $\text{Sym}[X] \rightarrow R$ defined by replacing p_n by $p_n(A)$. For instance, we would have

$$p_1 p_2[X/(1-q)] = p_1[X] p_2[X](1-q)^{-1}(1-q^2)^{-1}.$$

See [Hai01] for a reference.

The modified Macdonald polynomials [GHT99] will be denoted

$$H_\mu = t^{n(\mu)} J_\mu[X/(1-t^{-1}); q, t^{-1}] \in \text{Sym}[X]$$

where J_μ is the integral form of the Macdonald polynomial [Mac95], and

$$n(\mu) = \sum_i (i-1)\mu_i.$$

The operator $\nabla : \text{Sym}[X] \rightarrow \text{Sym}[X]$ is defined by

$$(3) \quad \nabla H_\mu = H_\mu[-1] H_\mu = (-1)^{|\mu|} q^{n(\mu')} t^{n(\mu)} H_\mu.$$

Note that our definition differs from the usual one from [BGHT99] by the sign $(-1)^{|\mu|}$. We also have the sequences of operators $B_r, C_r : \text{Sym}[X] \rightarrow \text{Sym}[X]$ given by the following formulas:

$$\begin{aligned} (B_r F)[X] &= F[X - (q-1)z^{-1}] \text{Exp}[-zX] \big|_{z^r}, \\ (C_r F)[X] &= -q^{1-r} F[X + (q^{-1}-1)z^{-1}] \text{Exp}[zX] \big|_{z^r}, \end{aligned}$$

where $\text{Exp}[X] = \sum_{n=0}^{\infty} h_n[X]$ is the plethystic exponential and $|_{z^r}$ denotes the operation of taking the coefficient of z^r of a Laurent power series. Our definition again differs from the one in [HMZ12] by a factor $(-1)^r$. For any composition α , let C_α denote the composition $C_{\alpha_1} \cdots C_{\alpha_l}$, and similarly for B_α .

Finally we denote by $x \mapsto \bar{x}$ the involutive automorphism of $\mathbb{Q}(q, t)$ obtained by sending q, t to q^{-1}, t^{-1} . We denote by ω the λ -ring automorphism of $\text{Sym}[X]$ obtained by sending X to $-X$ and by $\bar{\omega}$ its composition with $\bar{*}$, i.e.

$$(\omega F)[X] = F[-X], \quad (\bar{\omega} F)[X] = \bar{F}[-X].$$

2.2. Parking functions. We now recall the combinatorial background to state the Shuffle conjecture, for which we refer to Haglund's book [Hag08]. We consider the infinite grid on the top right quadrant of the plane. Its intersection points are denoted as (i, j) with $i, j \in \mathbb{Z}$. For each cell of the grid its coordinates (i, j) are the coordinates of the top right corner. Thus $i = 1, 2, \dots$ indexes the columns and $j = 1, 2, \dots$ indexes the rows. Let \mathbb{D} be the set of Dyck paths of all lengths. A Dyck path of length n is a grid path from $(0, 0)$ to (n, n) consisting of North and East steps that stays above the main diagonal $i = j$. For $\pi \in \mathbb{D}$ denote by $|\pi|$ its length n . For $\pi \in \mathbb{D}$, let

$$\text{area}(\pi) := \# \text{Area}(\pi), \quad \text{Area}(\pi) := \{(i, j) : i < j, (i, j) \text{ under } \pi\}.$$

This is the set of cells between the path and the diagonal. Let a_j denote the number of cells $(i, j) \in \text{Area}(\pi)$ in the row j . The *area sequence* is the sequence $a(\pi) = (a_1, a_2, \dots, a_n)$ and we have $\text{area}(\pi) = \sum_{j=1}^n a_n$.

Let $(x_1, 1), (x_2, 2), \dots, (x_n, n)$ be the cells immediately to the right of the North steps. The sequence $x(\pi) = (x_1, x_2, \dots, x_n)$ is called the *coarea sequence* and we have $a_j + x_j = j$ for all j .

We have the *dinv* statistic and the *Dinv* set defined by

$$\text{dinv}(\pi) := \# \text{Dinv}(\pi), \quad \text{Dinv}(\pi) := \text{Dinv}^0(\pi) \cup \text{Dinv}^1(\pi) =$$

$$\{(j, j') : 1 \leq j < j' \leq n, a_j = a_{j'}\} \cup \{(j, j') : 1 \leq j' < j \leq n, a_{j'} = a_j + 1\}.$$

For $(j, j') \in \text{Dinv}(\pi)$ we say that (x_j, j) *attacks* $(x_{j'}, j')$.

For any π , the set of *word parking functions* associated to π is defined by

$$\mathcal{WP}_\pi := \{w \in \mathbb{Z}_{>0}^n : w_j > w_{j+1} \text{ whenever } x_j = x_{j+1}\}.$$

In other words, the elements of \mathcal{WP}_π are n -tuples w of positive integers which, when written from bottom to top to the right of each North step, are strictly decreasing on cells such that one is on top of the other. For any w , let

$$\text{dinv}(\pi, w) := \# \text{Dinv}(\pi, w), \quad \text{Dinv}(\pi, w) := \{(j, j') \in \text{Dinv}(\pi) : w_j > w_{j'}\}.$$

We note that both of these conditions differ from the usual notation in which parking functions are expected to increase rather than decrease, and in which the inequalities are reversed in the definition of *dinv*. This corresponds to choosing the opposite total ordering on $\mathbb{Z}_{>0}$ everywhere, which does not affect the final answer, and is more convenient for the purposes of this paper.

Let us call $\alpha = (\alpha_1, \dots, \alpha_k) = \text{touch}(\pi)$ the *touch composition* of π if $\alpha_1, \dots, \alpha_k$ are the lengths of the gaps between the points where π touches the main diagonal starting at the lower left. Equivalently, $\sum_{i=1}^k \alpha_i = n$ and the numbers $1, 1 + \alpha_1, 1 + \alpha_1 + \alpha_2, \dots, 1 + \alpha_1 + \dots + \alpha_{k-1}$ are the positions of 0 in the area sequence $a(\pi)$.

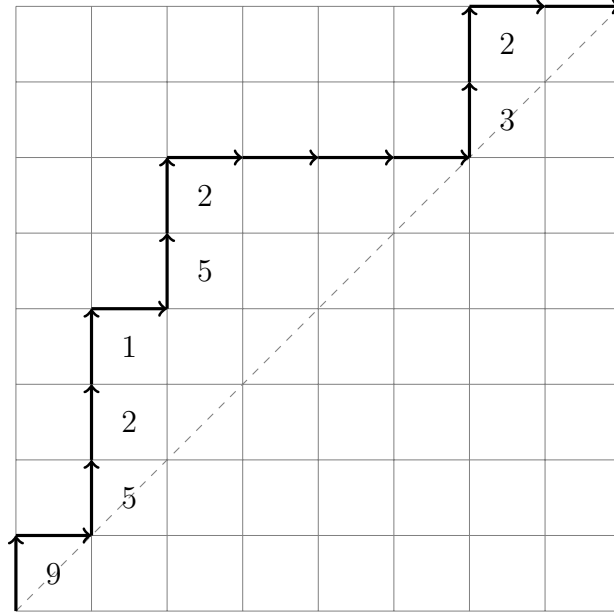


FIGURE 1. Example of a Dyck path of size 8.

Example 2.1. Let π be the following Dyck path of length 8 described in Figure 1. Then we have

$$\text{Area}(\pi) = \{(2, 3), (2, 4), (3, 4), (3, 5), (3, 6), (4, 5), (4, 6), (5, 6), (7, 8)\},$$

$$\text{Dinv}(\pi) = \{(1, 2), (1, 7), (2, 7), (3, 8), (4, 5)\} \cup \{(7, 3), (8, 4), (8, 5)\},$$

$$\text{touch}(\pi) = (1, 5, 2), \quad a(\pi) = (0, 0, 1, 2, 2, 3, 0, 1),$$

$$x(\pi) = (1, 2, 2, 2, 3, 3, 7, 7)$$

whence $\text{area}(\pi) = 9$, $\text{dinv}(\pi) = 5 + 3 = 8$. The labels shown above correspond to the vector $w = (9, 5, 2, 1, 5, 2, 3, 2)$, which we can see is an element of \mathcal{WP}_π because we have $5 > 2 > 1$, $5 > 2$, $3 > 2$. We then have

$$\text{Dinv}(\pi, w) = \{(1, 2), (1, 7), (2, 7)\} \cup \{(7, 3), (8, 4)\},$$

giving $\text{dinv}(\pi, w) = 5$.

2.3. The shuffle conjectures. For any infinite set of variables $X = \{x_1, x_2, \dots\}$, let $x_w = x_{w_1} \cdots x_{w_n}$. In this notation, the original shuffle conjecture [HHLRU05] states

Conjecture ([HHLRU05]). We have

$$(-1)^n \nabla e_n = \sum_{|\pi|=n} t^{\text{area}(\pi)} \sum_{w \in \mathcal{WP}_\pi} q^{\text{dinv}(\pi, w)} x_w.$$

In particular, the right hand side is symmetric in the x_i , and in q, t .

The stronger compositional shuffle conjecture [HMZ12] states

Conjecture ([HMZ12]). For any composition α , we have

$$(4) \quad (-1)^n \nabla C_\alpha(1) = \sum_{\text{touch}(\pi)=\alpha} t^{\text{area}(\pi)} \sum_{w \in \mathcal{WP}_\pi} q^{\text{dinv}(\pi, w)} x_w.$$

2.4. From (area, dinv) to (bounce, area'). In this paper, we will prove an equivalent version of this conjecture using different statistics, obtained by applying the (area, dinv) to (bounce, area') bijection, which can be found in [Hag08]. Our construction of this bijection is different and comes naturally from analysis of the attack relation. An important property of our construction is that it has a natural lift from Dyck paths to parking functions.

From any pair $\pi \in \mathbb{D}$, $w \in \mathcal{WP}_\pi$ we will obtain a pair $\pi' \in \mathbb{D}$, $w' \in \mathcal{WP}'_{\pi'}$ by a procedure described below. After the end of this section we will only work with π', w' , so we will drop the apostrophe.

The Dyck path w' is obtained as follows: sort the cells (x_j, j) in the *reading order*, i.e. in increasing order by the corresponding labels a_j , using the row index j to break ties. Equivalently, we read the cells by diagonals from bottom to top, and from left to right in each diagonal. For instance, for the path π from Example 2.1, the list would be

$$(5) \quad \{(1, 1), (2, 2), (7, 7), (2, 3), (7, 8), (2, 4), (3, 5), (3, 6)\}.$$

Let σ_j be the position of the cell (x_j, j) in this list. This defines a permutation $\sigma \in S_n$. In the example case, we would get

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 2 & 4 & 6 & 7 & 8 & 3 & 5 \end{pmatrix}$$

Now we observe that for each $j = 1, \dots, n$ the cell (x_j, j) attacks all the subsequent cells in the reading order whose position is before the position where we would place $(x_j, j+1)$, if it were an element of the list.

More precisely, there is a unique Dyck path π' for which

$$\text{Area}(\pi') = \sigma(\text{Dinv}(\pi)) = \{(\sigma_j, \sigma_{j'}) : (j, j') \in \text{Dinv}(\pi)\}.$$

The map $\pi \rightarrow \pi'$ is the desired bijection. To see the bijectivity one can either use [Hag08] or see Remark 2.1 below.

If π is the Dyck path from our example, then π' would be given by the path in Figure 2.

The above statistics can be translated into new statistics under this bijection. First, it is clear from the construction that $\text{dinv}(\pi) = \text{area}(\pi')$. We next explain how to calculate $\text{area}(\pi)$ from π' : for any path, we obtain a new Dyck path called the “bounce path” as follows: start at the origin $(0, 0)$, and begin moving North until contact is made with the first East step of π . Then start moving East until contacting the diagonal. Then move North until contacting the path again, and so on. Note that contacting the path means running into the left endpoint of an East step, but passing by the rightmost endpoint does not count, as illustrated below. The bounce path splits the main diagonal into the *bounce blocks*. We number the

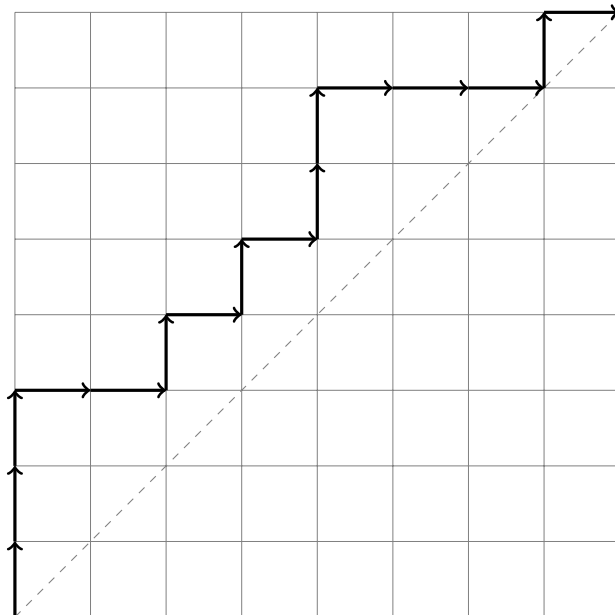


FIGURE 2. Image of the path from Figure 1 under the $(\text{area}, \text{dinv})$ to $(\text{bounce}, \text{area}')$ bijection.

bounce blocks starting from 0 and define the *bounce sequence* $b(\pi) = (b_1, b_2, \dots, b_n)$ in such a way that for any i the cell (i, i) belongs to the b_i -th block. We then define

$$\text{bounce}(\pi') := \sum_{i=1}^n b_i.$$

Another way to describe this construction is to say that $b_1 = 0$, $b_{i+1} \in \{b_i, b_i + 1\}$ and if i, i' are the smallest indices for which $b_i = c$ and $b_{i'} = c + 1$ for some c , then i' is the smallest index with $i' > i$ such that $(i, i') \notin \text{Area}(\pi')$. This description and $\text{Area}(\pi') = \sigma(\text{Dinv}(\pi))$ implies $b_{\sigma_i} = a_i$, hence $\text{bounce}(\pi') = \text{area}(\pi)$. See [Hag08] for an alternative treatment.

For the path π' above, the bounce path is shown in Figure 3 with the original path in gray. The bounce sequence is given by the numbers written under the diagonal. We have

$$b(\pi') = (0, 0, 0, 1, 1, 2, 2, 3), \quad \text{bounce}(\pi') = 9 = \text{area}(\pi).$$

Next, we show how to reconstruct $\text{touch}(\pi)$ from π' . For any path π' of length n , let l be the number of North steps from $(0, 0)$ until the first East step, which is the same as the length of the first bounce block. Let $\tilde{\pi}$ be the part of the path such that $\pi' = N^l E \tilde{\pi}$, the result of beginning with l North steps starting at the origin, followed by an East step, followed by the contents of $\tilde{\pi}$. Define numbers t_i by

$$t_i := \text{bounce}(N^{i+1} E N^{l-i} E \tilde{\pi}), \quad 0 \leq i \leq l.$$

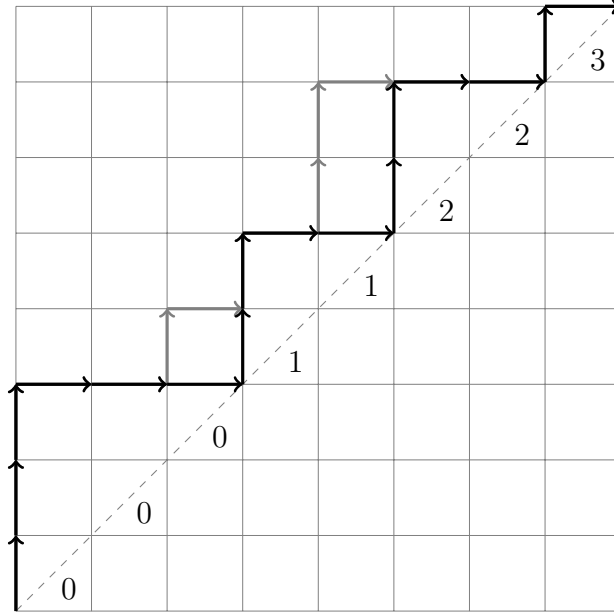


FIGURE 3. The bounce path of the path in Figure 2.

Note that the path $N^{i+1}EN^{l-i}E\tilde{\pi}$ has length $n + 1$ for each i , and we have $t_0 = n + \text{bounce}(\pi')$ and then t_i go down to $t_l = \text{bounce}(\pi')$. Define

$$\text{touch}'(\pi') := (t_0 - t_1, \dots, t_{l-1} - t_l).$$

Proposition 2.1. *For every Dyck path π*

$$\text{touch}'(\pi') = \text{touch}(\pi).$$

For instance, in the example above we would have $l = 3$,

$$(t_0, t_1, t_2, t_3) = (17, 16, 11, 9), \quad \text{touch}'(\pi') = \text{touch}(\pi) = (1, 5, 2).$$

Proof. Consider the i -th touch point (x, x) of π (we count the touch points starting from 0, i.e. $(0, 0)$ is the 0-th touch point.) It splits π into two parts: π_1 followed by π_2 . Construct a new path $\hat{\pi}$ of length $n + 1$ by taking a step North, then following a translated copy of π_2 , then taking a step East, then following a translated copy of π_1 . The new path has length $n + 1$ and its area is bigger than the area of π by $n - x$. The $(\text{area}, \text{dinv})$ to $(\text{bounce}, \text{area}')$ map applied to $\hat{\pi}$ gives precisely the path $N^{i+1}EN^{l-i}E\tilde{\pi}$. Thus we have

$$\text{area}(\hat{\pi}) = \text{bounce}(N^{i+1}EN^{l-i}E\tilde{\pi}) = t_i,$$

$$\text{area}(\hat{\pi}) = n - x + \text{area}(\pi) = n - x + \text{bounce}(\pi').$$

So the sizes of the gaps between the touch points of π are exactly the differences $t_{i-1} - t_i$. \square

Remark 2.1. The construction we have used in the proof above can also be used to prove the bijectivity of the $(\text{area}, \text{dinv})$ to $(\text{bounce}, \text{area}')$ map. Here is an idea of a proof. First, every Dyck path arises as $\hat{\pi}$ above for unique π and i . On

the other hand, every Dyck path can be uniquely written as $N^{i+1}EN^{l-i}E\tilde{\pi}$. Thus iterating the construction we obtain every Dyck path on each side of the (area, dinv) to (bounce, area') map in a unique way.

Having analysed the statistics associated to a Dyck path we turn to the analysis of what happens to word parking functions. The dinv statistic is straightforward. For any $w' \in \mathbb{Z}_{>0}^n$, let

$$\text{inv}(\pi', w') := \# \text{Inv}(\pi', w'), \quad \text{Inv}(\pi', w') := \{(i, j) \in \text{Area}(\pi'), w'_i > w'_j\},$$

so that

$$\text{Inv}(\pi', w') = \sigma(\text{Dinv}(\pi, w)), \quad w'_{\sigma_i} = w_i.$$

For the value of w from Example 2.1, we would have

$$w' = (9, 5, 3, 2, 2, 1, 5, 2), \quad \text{Inv}(\pi', w') = \{(1, 2), (1, 3), (2, 3), (3, 4), (5, 6)\}.$$

In particular, $\text{inv}(\pi', w') = \text{dinv}(\pi, w) = 5$.

Finally, we reconstruct the word parking function condition. A cell (i, j) is called a *corner* of π' if it is above the path, but both its Southern and Eastern neighbors are below the path. Denote the set of corners by $c(\pi')$. One can check that the corners of π' correspond to pairs of cells with one on top of the other in π . For instance, from our example we have $c(\pi') = \{(2, 4), (3, 5), (4, 6), (7, 8)\}$. More precisely, we have

$$c(\pi') := \{(\sigma_j, \sigma_{j+1}) : 1 \leq j < n, x_j = x_{j+1}\}.$$

We therefore define

$$(6) \quad \mathcal{WP}'_{\pi'} := \{w' \in \mathbb{Z}_{>0}^n : w'_i > w'_j \text{ for } (i, j) \in c(\pi')\},$$

so that the condition $w \in \mathcal{WP}_{\pi}$ is equivalent to $w' \in \mathcal{WP}'_{\pi'}$.

Putting this together, we have

Proposition 2.2. *For any composition α we have*

$$(7) \quad D_{\alpha}(q, t) = \sum_{\text{touch}'(\pi)=\alpha} t^{\text{bounce}(\pi)} \sum_{w \in \mathcal{WP}'_{\pi}} q^{\text{inv}(\pi, w)}$$

where $D_{\alpha}(q, t)$ is the right hand side of (4).

3. CHARACTERISTIC FUNCTIONS OF DYCK PATHS

3.1. Simple characteristic function. We are going to study the summand in $D_{\alpha}(q, t)$ as a function of π . It is convenient to first introduce a simpler object where we drop the assumption $w \in \mathcal{WP}'_{\pi}$ and instead sum over all labellings. Given a Dyck path of length n , define $\chi(w) \in \text{Sym}[X]$ as follows:

Definition 3.1.

$$\chi(\pi) := \sum_{w \in \mathbb{Z}_{>0}^n} q^{\text{inv}(\pi, w)} x_w.$$

If $i < j$ and (i, j) is under π , i.e. $(i, j) \in \text{Area}(\pi)$ we say that i and j *attack* each other. The proof of Lemma 10.2 from [HHL05] applies in our case and we obtain

Proposition 3.1. *The expression for $\chi(\pi)$ above is symmetric in the variables x_1, x_2, x_3, \dots , so that Definition 3.1 correctly defines an element of $\text{Sym}[X]$.*

Another way to formulate this property is as follows: for a composition $c_1 + c_2 + \dots + c_k = n$ consider the multiset $M_c = 1^{c_1} 2^{c_2} \dots k^{c_k}$. Consider the sum

$$\sum_{w \text{ a permutation of } M_c} q^{\text{inv}(\pi, w)}.$$

Proposition 3.1 simply says that this sum does not depend on the order of the numbers c_1, c_2, \dots, c_k , or equivalently on the linear order on the set of labels. If λ is the partition with components c_1, c_2, \dots, c_k , then this sum computes the coefficient of the monomial symmetric function m_λ in $\chi(\pi)$, so we have (set $h_c = h_{c_1} \dots h_{c_k}$)

$$(8) \quad (\chi(\pi), h_c) = \sum_{w \text{ a permutation of } M_c} q^{\text{inv}(\pi, w)}.$$

We list here a few properties of χ so that the reader has a feeling of what kind of object it is.

For a Dyck path π denote by π^{op} the reversed Dyck path, i.e. the path obtained by replacing each North step by East step and each East step by North step and reversing the order of steps. Reversing also the order of the components of c in (8) we see

Proposition 3.2.

$$\chi(\pi) = \chi(\pi^{op}).$$

Using Chapter 4 of [HHL05] (which is in fact a sort of inclusion-exclusion principle) we obtain

$$(\chi(\pi), e_c) = \sum_{w \text{ a permutation of } M_c} q^{\text{inv}'(\pi, w)},$$

where $\text{inv}'(\pi, w)$ is the number of non-strict inversions of w under the path,

$$\text{inv}'(\pi, w) := \# \{(i, j) \in \text{Area}(\pi), w_i \geq w_j\}.$$

Thus we have

$$(\chi(\pi)[-X], h_c[X]) = (-1)^{|\pi|} \sum_{w \text{ a permutation of } M_c} q^{\text{inv}'(\pi, w)},$$

or, reversing the order of labels,

$$(\chi(w)[-X], h_c[X]) = (-1)^{|\pi|} \sum_{\tilde{w} \text{ a permutation of } M_c} q^{\text{area}(\pi) - \text{inv}(\pi, w)},$$

which implies

Proposition 3.3.

$$\bar{\omega}\chi(\pi) = (-1)^{|\pi|} q^{-\text{area}(\pi)} \chi(\pi).$$

Chapter 4 and Lemma 5.1 from [HHL05] also apply in a straightforward way to our case and we obtain

Proposition 3.4.

$$\chi(\pi)[(q-1)X] = (q-1)^{|\pi|} \sum_{w \in \mathbb{Z}_{>0}^{|\pi|} \text{no attack}} q^{\text{inv}(\pi, w)} x_w,$$

where “no attack” means that the summation is only over vectors w such that $w_i \neq w_j$ for $(i, j) \in \text{Area}(\pi)$.

3.2. Weighted characteristic function. To study the summand of $D_\alpha(q, t)$ in (7) as a function of π we introduce a more general characteristic function. Given a function $\text{wt} : c(\pi) \rightarrow R$ on the set of corners of some Dyck path π of size n , let

$$(9) \quad \chi(\pi, \text{wt}) := \sum_{w \in \mathbb{Z}_{>0}^n} q^{\text{inv}(\pi, w)} \left(\prod_{(i, j) \in c(w), w_i \leq w_j} \text{wt}(i, j) \right) x_w,$$

so in particular (7) becomes

$$D_\alpha(q, t) = \sum_{\text{touch}'(\pi) = \alpha} t^{\text{bounce}(\pi)} \chi(\pi, 0).$$

For a constant function $\text{wt} = 1$ we recover the simpler characteristic function

$$(10) \quad \chi(\pi, 1) = \chi(\pi).$$

It turns out that we can express the weighted characteristic function $\chi(\pi, \text{wt})$ in terms the unweighted one evaluated at different paths. In particular this implies that $\chi(\pi, \text{wt})$ is symmetric too.

Proposition 3.5. *We have that $\chi(\pi, \text{wt})$ is symmetric in the x_i variables, and so defines an element of $\text{Sym}[X]$.*

Proof. Let π be a Dyck path, and let $(i, j) \in c(\pi)$ be one of its corners. We denote by wt_1 the weight on π which is obtained from wt by setting the weight of (i, j) to 1. Let π' be the Dyck path obtained from π by turning the corner inside out, in other words the Dyck path of smallest area which is both above π and above (i, j) . Let wt_2 be the weight on π' which coincides with wt on all corners of π' which are also corners of π and is 1 on other corners. We claim that

$$(11) \quad \chi(\pi, \text{wt}) = \frac{q\text{wt}(i, j) - 1}{q - 1} \chi(\pi, \text{wt}_1) + \frac{1 - \text{wt}(i, j)}{q - 1} \chi(\pi', \text{wt}_2).$$

To see this, notice that if we group the terms on the right hand side, then both sides may be written as a sum over vectors $w \in \mathbb{Z}_{>0}$. Split both sums according to terms in which $w_i > w_j$ resulting in an additional factor of q , or $w_i \leq w_j$ resulting in an additional weight factor. It is easy to check that both sums agree on both the left and right sides.

The result now follows because we may recursively express any $\chi(\pi, \text{wt})$ in terms of $\chi(\pi)$, which we have already remarked is symmetric. □

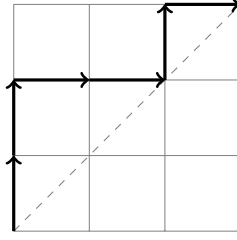


FIGURE 4.

Example 3.1. In particular, we can use this to extract $\chi(\pi, 0)$ from $\chi(\pi', 1)$ for all π' . If $S \subset c(\pi)$ is any subset of the set of corners, let $\pi_S \in \mathbb{D}$ denote the path obtained by flipping the corners that are in S . Then equation (11) implies that

$$(12) \quad \chi(\pi, 0) = (1 - q)^{-|c(\pi)|} \sum_{S \subset c(\pi)} (-1)^{|S|} \chi(\pi_S, 1).$$

For instance, let π be the Dyck path in Figure 4. Then setting $x_i = 0$ for $i > 3$ reduces formula (9) to a finite sum over 27 terms, from which we can deduce that

$$\chi(\pi) = m_3 + (2 + q)m_{21} + (3 + 3q)m_{111} = s_3 + (1 + q)s_{21} + qs_{111}.$$

Similarly, if $\pi' = \pi_{\{(1,2)\}}$ we have

$$\chi(\pi') = s_3 + 2qs_{21} + q^2s_{111}.$$

By formula (12), we obtain

$$\chi(\pi, 0) = (1 - q)^{-1} (\chi(\pi) - \chi(\pi')) = s_{21} + qs_{111}.$$

Example 3.2. We can check that the Dyck path from Example 3.1 is the unique one satisfying $\text{touch}'(\pi) = (1, 2)$, and that $\text{bounce}(\pi) = 1$. Therefore, using the calculation that followed we have that

$$D_{(2,1)}(q, t) = t\chi(\pi, 0) = ts_{21} + qts_{111}$$

which can be seen to agree with $\nabla C_1 C_2(1)$.

Example 3.3. Though we will not need it, this weighted characteristic function can be used to describe an interesting reformulation of the formula for the modified Macdonald polynomial given in [HHL05]. Let $\mu = (\mu_1 \geq \mu_2 \geq \dots \geq \mu_l)$ be a partition of size n . Let us list the cells of μ in the reading order:

$$(l, 1), (l, 2), \dots, (l, \mu_l), (l-1, 1), \dots, (l-1, \mu_{l-1}), \dots, (1, 1), \dots, (1, \mu_1).$$

Denote the m -th cell in this list by (i_m, j_m) .

We say that a cell (i, j) attacks all cells which are after (i, j) and before $(i-1, j)$. Thus (i, j) attacks precisely $\mu_i - 1$ following cells if $i > 1$ and all following cells if $i = 1$. Next construct a Dyck path π_μ of length n in such a way that (m_1, m_2) with $m_1 < m_2$ is under the path if and only if (i_{m_1}, j_{m_1}) attacks (i_{m_2}, j_{m_2}) . More specifically, the path begins with μ_l North steps, then it has μ_l pairs of steps East-North, then $\mu_{l-1} - \mu_l$ North steps followed by μ_{l-1} East-North pairs and so on until we reach the point $(n - \mu_1, n)$. We complete the path by performing μ_1 East steps.

Note that the corners of π_μ precisely correspond to the pairs of cells $(i, j), (i - 1, j)$. We set the weight of such a corner to $q^{\text{arm}(i,j)} t^{-1 - \text{leg}(i,j)}$ and denote the weight function thus obtained by wt_μ . Note that in our convention for $\chi(\pi, \text{wt})$ we should count non-inversions in the corners, while in [HHL05] they count “descents,” which translates to counting inversions in the corners. Taking this into account, we obtain a translation of their Theorem 2.2:

$$H_\mu = q^{-n(\mu') + \binom{\mu_1}{2}} t^{n(\mu)} \chi(\pi_\mu, \text{wt}_\mu).$$

4. RAISING AND LOWERING OPERATORS

Now let $\mathbb{D}_{k,n}$ be the set of Dyck paths from $(0, k)$ to (n, n) , which we will call partial Dyck paths, and let \mathbb{D}_k be their union over all n . For $\pi \in \mathbb{D}_{k,n}$ let $|\pi| = n - k$ denote the number of North steps. Unlike \mathbb{D} , the union of the sets \mathbb{D}_k over all k is closed under the operation of adding a North or East step to the beginning of the path, and any Dyck path may be created in such a way starting with the empty path in \mathbb{D}_0 . This is the set of paths that we will develop a recursion for. More precisely, we will define an extension of the function χ to a map from \mathbb{D}_k to a new vector space V_k , and prove that certain operators on these vector spaces commute with adding North and East steps.

Given a polynomial P depending on variables u, v define

$$\begin{aligned} (\Delta_{uv} P)(u, v) &= \frac{(q - 1)vP(u, v) + (v - qu)P(v, u)}{v - u}, \\ (\Delta_{uv}^* P)(u, v) &= \frac{(q - 1)uP(u, v) + (v - qu)P(v, u)}{v - u}. \end{aligned}$$

We can easily check that $\Delta_{uv}^* = q\Delta_{uv}^{-1}$. We can recognize these operators as a simple modification of Demazure-Lusztig operators. The following can be checked by direct computation:

Proposition 4.1. *We have the following relations:*

$$\begin{aligned} (\Delta_{uv} - q)(\Delta_{uv} + 1) &= 0, & (\Delta_{uv}^* - 1)(\Delta_{uv}^* + q) &= 0, \\ \Delta_{uv}\Delta_{vw}\Delta_{uv} &= \Delta_{vw}\Delta_{uv}\Delta_{vw}, & \Delta_{uv}^*\Delta_{vw}^*\Delta_{uv}^* &= \Delta_{vw}^*\Delta_{uv}^*\Delta_{vw}^*. \end{aligned}$$

Definition 4.1. Let $V_k = \text{Sym}[X] \otimes \mathbb{Q}[y_1, y_2, \dots, y_k]$, and let

$$T_i = \Delta_{y_i y_{i+1}}^* : V_k \rightarrow V_k, \quad i = 1, \dots, k - 1.$$

Define operators

$$d_+ : V_k \rightarrow V_{k+1}, \quad d_- : V_k \rightarrow V_{k-1}$$

by

$$(13) \quad (d_+ F)[X] = T_1 T_2 \cdots T_k (F[X + (q - 1)y_{k+1}]),$$

for $F \in V_k$, and

$$(14) \quad d_-(y_k^i F) = -B_{i+1} F$$

when F does not depend on y_k .

We now claim the following theorem:

Theorem 4.2. *For any Dyck path π of size n , let $\epsilon_1 \cdots \epsilon_{2n}$ denote the corresponding sequence of plus and minus symbols where a plus denotes an east step, and a minus denotes a north step reading π from bottom left to top right. Then*

$$\chi(\pi) = d_{\epsilon_1} \cdots d_{\epsilon_{2n}}(1)$$

as an element of $V_0 = \text{Sym}[X]$.

Example 4.1. Let π be the Dyck path from Example 3.1. We have that

$$\begin{aligned} d_- d_- d_+ d_+ d_- d_+(1) &= d_- d_- d_+ d_+ d_-(1) = d_- d_- d_+ d_+(s_1) = \\ d_- d_- d_+(s_1 + (q-1)y_1) &= d_- d_-(s_1 + (q-1)(y_1 + y_2)) = \\ d_-(s_2 + s_{11} + (q-1)s_1 y_1) &= s_3 + (1+q)s_{21} + qs_{111}, \end{aligned}$$

which agrees with the value calculated for $\chi(\pi)$.

Combining this result with equation (12) implies the following:

Corollary 4.3. *The following procedure computes $\chi(\pi, 0)$: start with $1 \in \text{Sym}[X] = V_0$, follow the path from right to left applying $\frac{1}{q-1}[d_-, d_+]$ for each corner of w , and d_- (d_+) for each North (resp. East) step which is not a side of a corner.*

Remark 4.1. Before we proceed to the proof of Theorem 4.2, we would like to explain why we expected such a result to hold and how we obtained it. First note that the number of Dyck paths of length n is given by the Catalan number $C_n = \frac{1}{n+1} \binom{2n}{n}$ which grows exponentially with n . The dimension of the degree n part of $\text{Sym}[X]$ is the number of partitions of size n , which grows subexponentially. For instance, for $n = 3$ we have 5 Dyck paths, but only 3 partitions. Thus there must be linear dependences between different $\chi(\pi)$. Now fix a partial Dyck path $\pi_1 \in \mathbb{D}_{k,n}$. For each partial Dyck path $\pi_2 \in \mathbb{D}_{k,n'}$ we can reflect π_2 and concatenate it with π_1 to obtain a full Dyck path $\pi_2^{op} \pi_1$ of length $n + n' - k$. Then we take its character $\chi(\pi_2^{op} \pi_1)$. We keep n , π_1 fixed and vary n' , π_2 . Thus we obtain a map $\varphi_{\pi_1} : \mathbb{D}_k \rightarrow \text{Sym}[X]$. The map $\pi_1 \rightarrow \varphi_{\pi_1}$ is a map from \mathbb{D}_k to the vector space of maps from \mathbb{D}_k to $\text{Sym}[X]$, which is very high dimensional, because both the set \mathbb{D}_k is infinite and $\text{Sym}[X]$ is infinite dimensional. A priori it could be the case that the images of the elements of $\mathbb{D}_{k,n}$ in $\text{Maps}(\mathbb{D}_k, \text{Sym}[X])$ are linearly independent. But computer experiments convinced us that it is not so, that there should be a vector space $V_{k,n}$ whose dimension is generally smaller than the size of $\mathbb{D}_{k,n}$, so that we have a commutative diagram

$$\begin{array}{ccc} \mathbb{D}_{k,n} & \xrightarrow{\chi_{k,n}} & V_{k,n} \\ \downarrow \varphi & \swarrow & \\ \text{Maps}(\mathbb{D}_k, \text{Sym}[X]) & & \end{array}$$

We then guessed that $V_{k,n}$ should be the degree n part of $V_k = \text{Sym}[X] \otimes \mathbb{Q}[y_1, \dots, y_k]$, and from that conjectured a definition of $\chi_k : \mathbb{D}_k \rightarrow V_k$ defined below, and verified on examples that partial Dyck paths that are linearly dependent after applying $\chi_{k,n}$ satisfy the same linear dependence after applying φ . Once this was established, the

computation of the operators d_- , d_+ and proof of Theorem 4.2 turned out to be relatively straightforward.

The proof of Theorem 4.2 will be divided in several parts.

4.1. Characteristic functions of partial Dyck paths. The following definition is motivated by Proposition 3.4. Let $\pi \in \mathbb{D}_{k,n}$. Let $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_k) \in \mathbb{Z}_{>0}$ be a tuple of distinct numbers. The elements of $\text{Im}(\sigma) \subset \mathbb{Z}_{>0}$ will be called *special*. Let

$$U_{\pi,\sigma} = \{w \in \mathbb{Z}_{>0}^n : w_i = \sigma_i \text{ for } i \leq k, w_i \neq w_j \text{ for } (i,j) \in \text{Area}(\pi)\}.$$

The second condition on w is the “no attack” condition as before. The first condition says that we put the special labels in the positions $1, 2, \dots, k$ as prescribed by σ . Let

$$(15) \quad \chi'_\sigma(\pi) = \sum_{w \in U_{\pi,\sigma}} q^{\text{inv}(\pi,w)} z_w.$$

Here we use variables z_1, z_2, \dots .

Suppose σ is a permutation, i.e $\sigma_i \leq k$ for all i . Set $z_i = y_i$ for $i \leq k$ and $z_i = x_{i-k}$ for $i > k$. We denote

$$\chi'_k(\pi) = \chi'_{(1,2,\dots,k)}(\pi).$$

Let us group the summands in (15) according to the positions of special labels. More precisely, let $S \subset \{1, \dots, n\}$ such that $\{1, \dots, k\} \subset S$ and $w^S : S \rightarrow \{1, \dots, k\}$ such that $w_i^S = \sigma_i$ for $i = 1, 2, \dots, k$ and $w_i \neq w_j$ for $i, j \in S$, $(i, j) \in \text{Area}(\pi)$. Set

$$U_{\pi,\sigma}^{S,w^S} := \{w \in U_{\pi,\sigma} : w_i = w_i^S \text{ for } i \in S, w_i > k \text{ for } i \notin S\},$$

$$\Sigma_{\pi,\sigma}^{S,w^S} := \sum_{w \in U_{\pi,\sigma}^{S,w^S}} q^{\text{inv}(\pi,w)} x_w, \quad \chi'_\sigma(\pi) = \sum_{S, w^S} \Sigma_{\pi,\sigma}^{S,w^S}.$$

Let $m_1 < m_2 < \dots < m_r$ be all the positions not in S . Let π_S be the unique Dyck path of length r such that $(i, j) \in \text{Area}(\pi_S)$ if and only if $m_i, m_j \in \text{Area}(\pi)$. We have

$$\Sigma_{\pi,\sigma}^{S,w^S} = q^A \prod_{i \in S} y_{w_i} \sum_{w \in \mathbb{Z}_{>0}^r \text{no attack}} q^{\text{inv}(\pi_S,w)} x_w,$$

where

$$A = \#\{(i, j) \in \text{Area}(\pi) : (i \in S, j \in S, w_i^S > w_j^S) \text{ or } (i \notin S, j \in S)\}.$$

By Proposition 3.4 we have

$$(16) \quad \Sigma_{\pi,\sigma}^{S,w^S} = q^A (q-1)^{|S|-n} \chi(\pi^S) [(q-1)X] \prod_{i \in S} y_{w_i}.$$

In particular, $\chi_\sigma(\pi)$ is a symmetric function in x_1, x_2, \dots and it makes sense to define

$$(17) \quad \chi_\sigma(\pi)[X] := \frac{1}{y_1 y_2 \dots y_k} (q-1)^{|\pi|} \chi'_\sigma(\pi) \left[\frac{X}{q-1} \right] \in V_k, \quad \chi_k(\pi) := \chi_{\text{Id}_k}(\pi).$$

Remark 4.2. The identity (16) also implies that the coefficients of $\chi_\sigma(\pi)[X]$ are polynomials in q and gives a way to express χ_σ in terms of the characteristic functions $\chi(\pi_S)$ for all S .

For $k = 0$ we recover $\chi(\pi)$:

$$\chi_0(\pi) = \chi(\pi) \quad (\pi \in \mathbb{D}_0 = \mathbb{D}).$$

Thus, it suffices to prove that

$$(18) \quad \chi_{k+1}(E\pi) = d_+\chi_k(\pi), \quad \chi_{k-1}(N\pi) = d_-\chi_k(\pi) \quad (\pi \in \mathbb{D}_k).$$

4.2. Raising operator. We begin with the first case. Let $\pi \in \mathbb{D}_{k,n}$ so that $E\pi \in \mathbb{D}_{k+1,n+1}$, and we need to express $\chi_{k+1}(E\pi)$ in terms of $\chi_k(\pi)$. Let σ be the following sequence:

$$\sigma = (k+1, 1, 2, \dots, k).$$

Then we have a bijection $f : U_{\pi, \text{Id}_k} \rightarrow U_{E\pi, \sigma}$ obtained by sending

$$w = (1, 2, \dots, k, w_{k+1}, \dots, w_n)$$

to

$$f(w) := (k+1, 1, 2, \dots, k, w_{k+1}, \dots, w_n).$$

This is possible because 1 does not attack $k+1$ in $E\pi$. We clearly have $\text{inv}(E\pi, f(w)) = \text{inv}(\pi, w) + k$, which implies

$$\chi'_\sigma(E\pi) = z_{k+1}q^k\chi'_k(\pi),$$

where both sides are written in terms of the variables z_i . When we pass to the variables x_i, y_i on the left we have

$$(z_1, z_2, \dots) = (y_1, y_2, \dots, y_{k+1}, x_1, x_2, \dots),$$

but on the right we have

$$(z_1, z_2, \dots) = (y_1, y_2, \dots, y_k, x_1, x_2, \dots),$$

thus we need to perform the substitution $X = y_{k+1} + X$:

$$\chi'_\sigma(E\pi)[X] = y_{k+1}q^k\chi'_k(\pi)[X + y_{k+1}],$$

Performing the transformation (17) we obtain

$$(19) \quad \chi_\sigma(E\pi) = q^k\chi_k(\pi)[X + (q-1)y_{k+1}].$$

To finish the computation we need to relate $\chi_{k+1} = \chi_{\text{Id}_{k+1}}$ and χ_σ . We first note that σ can be obtained from Id_{k+1} by successively swapping neighboring labels. Let $\sigma^{(1)} = \text{Id}_{k+1}$ and

$$\sigma^{(i)} = (i, 1, 2, \dots, i-1, i+1, \dots, k+1) \quad (i = 2, 3, \dots, k+1),$$

so that $\sigma = \sigma^{(k+1)}$. It is clear that $\sigma^{(i+1)}$ can be obtained from $\sigma^{(i)}$ by interchanging the labels i and $i+1$.

We show below (Proposition 4.4) that this kind of interchange is controlled by the operator $\Delta_{y_i, y_{i+1}}$:

$$(20) \quad \chi_{\sigma^{(i+1)}}(E\pi) = \Delta_{y_i, y_{i+1}}\chi_{\sigma^{(i)}}(E\pi).$$

This implies

$$\chi_\sigma(E\pi) = \Delta_{y_{k-1}, y_k} \cdots \Delta_{y_1, y_2}\chi_{k+1}(E\pi).$$

When we insert this equation into (19), we arrive at

$$\chi_{k+1}(E\pi) = T_1 \cdots T_k (\chi_k(\pi) [X + (q-1)y_{k+1}]) = d_+ \chi_k(\pi).$$

4.3. Swapping operators.

Proposition 4.4. *For any $w \in \mathbb{D}_k$, σ as above and m special suppose that $m+1$ is not special or $\sigma^{-1}(m) < \sigma^{-1}(m+1)$. Then we have*

$$\chi'_{\tau_m \sigma}(w) = \Delta_{z_m, z_{m+1}} \chi'_\sigma,$$

where τ_m is the transposition $m \leftrightarrow m+1$, $(\tau_m \sigma)_i = \tau_m(\sigma_i)$ for $i = 1, \dots, k$.

Proof. We decompose both sides as follows. For any $w \in U_{\pi, \sigma}$ let $S(w)$ be the set of indices j where $w_j \in \{m, m+1\}$. For $w, w' \in U_{\pi, \sigma}$ write $w \sim w'$ if $S(w) = S(w')$ and $w_i = w'_i$ for all $i \notin S(w)$. This defines an equivalence relation on $U_{\pi, \sigma}$. The sum (15) is then decomposed as follows:

$$(21) \quad \chi'_\sigma(\pi) = \sum_{[w] \in U_{\pi, \sigma} / \sim} q^{\text{inv}_1(\pi, w)} \prod_{i \notin S} z_{w_i} \sum_{w' \sim w} a(w'),$$

where

$$\text{inv}_1(\pi, w) = \#\{(i, j) \in \text{Area}(\pi) : w_i > w_j, i \notin S(w) \text{ or } j \notin S(w)\},$$

which does not depend on the choice of a representative w in the equivalence class $[w]$, and

$$a(w) = q^{\text{inv}_2(\pi, w)} \prod_{i \in S} z_{w_i},$$

$$\text{inv}_2(\pi, w) = \#\{(i, j) \in \text{Area}(\pi) : w_i > w_j, i, j \in S(w)\}.$$

Let $f : U_{\pi, \sigma} \rightarrow U_{\pi, \tau_m \sigma}$ be the bijection defined by $f(w)_i = \tau_m(w_i)$. This bijection respects the equivalence relation \sim and we have $S(f(w)) = S(w)$. Moreover, we have $\text{inv}_1(\pi, w) = \text{inv}_1(\pi, f(w))$. We now make the stronger claim that for any $w \in U_{\pi, \sigma}$

$$(22) \quad \sum_{w' \sim f(w)} a(w') = \Delta_{z_m, z_{m+1}} \sum_{w' \sim w} a(w')$$

which would imply the statement by summing over all equivalence classes.

For each $w \in U_{\pi, \sigma}$ the set $S(w)$ is decomposed into a disjoint union of *runs*, i.e. subsets

$$R = \{j_1, \dots, j_l\} \subset \{1, \dots, n\}, \quad j_1 < \dots < j_l$$

such that in each run j_a attacks j_{a+1} for all a and elements of different runs do not attack each other. Because of the non attacking condition, the labels w_{j_a} must alternate between $m, m+1$ and j_a does not attack j_{a+2} . Thus to fix w in each equivalence class it is enough to fix w_{j_1} for each run. Suppose the runs of $S(w)$ have lengths l_1, l_2, \dots, l_r and the first values of w in each run are c_1, c_2, \dots, c_r respectively.

With this information $a(w)$ can be computed as follows:

$$a(w) = \prod_{i=1}^r a(l_i, c_i),$$

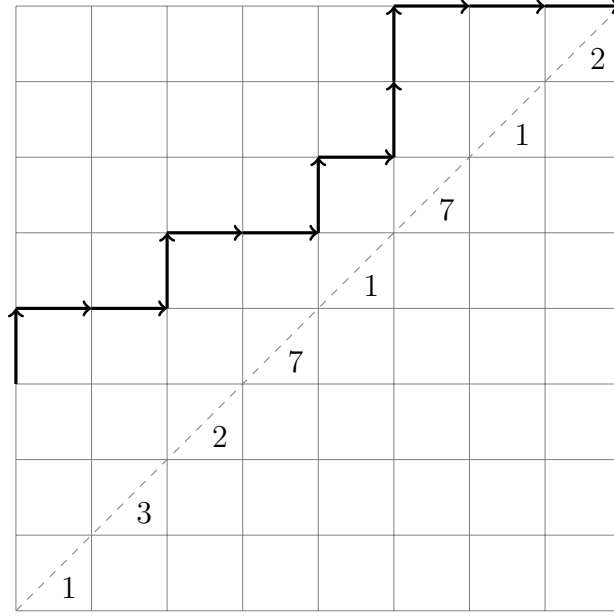


FIGURE 5.

where

$$a(l, c) := \begin{cases} q^{l'-1} z_m^{l'} z_{m+1}^{l'} & l = 2l', c = m \\ q^{l'} z_m^{l'+1} z_{m+1}^{l'} & l = 2l' + 1, c = m \\ q^{l'} z_m^{l'} z_{m+1}^{l'+1} & l = 2l', c = m + 1 \\ q^{l'} z_m^{l'} z_{m+1}^{l'+1} & l = 2l' + 1, c = m + 1. \end{cases}$$

For instance, let $k = 3$ and π be the Dyck path in Figure 5, and let

$$w = (1, 3, 2, 7, 1, 7, 1, 2) \in U_{\pi, (132)}.$$

Let $m = 1$. Then we have $S(w) = \{1, 3, 5, 7, 8\}$, which decomposes into two runs $\{1, 3, 5\}$ and $\{7, 8\}$. So we have $r = 2$, $(l_1, l_2) = (3, 2)$, $(c_1, c_2) = (1, 1)$ and we obtain

$$a(w) = a(3, 1)a(2, 1) = qz_1^2 z_2 z_1 z_2 = qz_1^3 z_2^2.$$

Note that by the assumption on σ we have $c_1 = m$, while c_i can take arbitrary values $\{m, m + 1\}$ for $i > 1$. This implies

$$\sum_{w' \sim w} a(w') = a(l_1, m) \prod_{i=2}^r (a(l_i, m) + a(l_i, m + 1)).$$

On the other hand we have

$$\sum_{w' \sim f(w)} a(w') = \sum_{w' \sim w} a(f(w')) = a(l_1, m + 1) \prod_{i=2}^r (a(l_i, m) + a(l_i, m + 1)).$$

Now notice that for all l the sum $a(l, m) + a(l, m + 1)$ is symmetric in z_m, z_{m+1} . The operator $\Delta_{z_m, z_{m+1}}$ commutes with multiplication by symmetric functions and

satisfies

$$\Delta_{z_m, z_{m+1}}(a(l, m)) = a(l, m + 1).$$

This establishes (22) and the proof is complete. \square

Remark 4.3. The arguments used in the proof can be used to show that in the case when $m, m + 1$ are both not special the function $\chi'_\sigma(\pi)$ is symmetric in z_m, z_{m+1} . In particular, we can obtain a direct proof of the fact that χ'_σ is symmetric in the variables $z_m, z_{m+1}, z_{m+2}, \dots$ for $i = \max(\sigma) + 1$, without use of Proposition 3.4.

4.4. Lowering operator. We now turn to the remaining identity $\chi_{k-1}(N\pi) = d_- \chi_k(\pi)$. Assume $\pi \in \mathbb{D}_{k,n}$, so that $N\pi \in \mathbb{D}_{k-1,n}$. We observe that

$$\chi'_{k-1}(N\pi)[X + y_k] = \sum_{r \geq 0} \chi'_{k,r}(\pi)[X],$$

where

$$\chi'_{k,r}(\pi) = \chi'_\sigma(\pi), \quad \sigma = (1, 2, \dots, k-1, k+r)$$

and to get to the second equality we have summed over all possible values of $r = w_k - k$ that do not result in an attack. It is convenient to set $x_0 = y_k$. Using Proposition 4.4 we can characterize $\chi'_{k,r}(\pi)$ by

$$(23) \quad \chi'_{k,0}(\pi) = \chi'_k(\pi), \quad \chi'_{k,r+1}(\pi) = \Delta_{x_r, x_{r+1}} \chi'_{k,r}(\pi) \quad (r \geq 0).$$

Now notice that there is a unique expansion

$$\chi'_k(\pi)[X] = \sum_{j \geq 1} y_k^j g_j(\pi)[X + y_k], \quad g_j(\pi) \in V_{k-1}.$$

The advantage over the more obvious expansion in powers of y_k is that each coefficient $g_j[X + y_k]$ is symmetric in the variables y_k, x_1, \dots . As a result, we have that

$$\chi'_{k,r}(\pi)[X] = \Delta_{x_{r-1}, x_r} \cdots \Delta_{x_2, x_1} \Delta_{y_k, x_1} \sum_{i \geq 1} y_k^i g_i(\pi)[X + y_k] = \sum_{i \geq 1} f_{i,r} g_i(\pi)[X + y_k]$$

where

$$f_{i,r} = \Delta_{x_{r-1}, x_r} \cdots \Delta_{x_1, x_2} \Delta_{y_k, x_1} (y_k^i) \quad (i \geq 1, r \geq 0)$$

The extra symmetry in the y_k variable is used to pass Δ_{y_k, x_1} by multiplication by $g_i(\pi)[X + y_k]$.

Now we need an explicit formula for $f_{i,r}$:

Proposition 4.5. *Denote $X_r = y_k + x_1 + \cdots + x_r$, $X_{-1} = 0$. We have*

$$f_{i,r} = \frac{h_i[(1-q)X_r] - h_i[(1-q)X_{r-1}]}{1-q}.$$

Proof. Denote the right hand side by $f'_{i,r}$. The proof goes by induction on r . For $r = 0$ both sides are equal to y_k^i . Thus it is enough to show that

$$(24) \quad \Delta_{x_r, x_{r+1}}(f'_{i,r}) = f'_{i,r+1}.$$

Use $X_r = X_{r-1} + x_r$ to write $f'_{i,r}$ as follows:

$$(25) \quad f'_{i,r} = \sum_{j=1}^i x_r^j h_{i-j}[(1-q)X_{r-1}] = x_r h_{i-1}[(1-q)X_{r-1} + x_r].$$

Now X_{r-1} does not contain the variables x_r, x_{r+1} , so we have

$$\Delta_{x_r, x_{r+1}}(f'_{i,r}) = \sum_{j=1}^i h_{i-j}[(1-q)X_{r-1}] \Delta_{x_r, x_{r+1}} x_r^j.$$

Using the formula

$$\Delta_{x_r, x_{r+1}} x_r^j = x_{r+1} h_{j-1}[(1-q)x_r + x_{r+1}],$$

which is straightforward to check, we can evaluate

$$\begin{aligned} \Delta_{x_r, x_{r+1}} f'_{i,r} &= x_{r+1} \sum_{j=1}^i h_{j-1}[(1-q)x_r + x_{r+1}] h_{i-j}[(1-q)X_{r-1}] \\ &= x_{r+1} h_{i-1}[(1-q)X_r + x_{r+1}], \end{aligned}$$

which matches $f'_{i,r+1}$ by (25). □

Now, if we sum over all r , we obtain

$$(26) \quad \sum_{r \geq 0} f_{i,r} = (1-q)^{-1} h_i[(1-q)(X + y_k)].$$

Thus

$$\chi'_{k-1}(N\pi)[X + y_k] = (1-q)^{-1} \sum_{i \geq 1} h_i[(1-q)(X + y_k)] g_i(\pi)[X + y_k].$$

This implies

$$(27) \quad \chi_{k-1}(N\pi)[X] = -\frac{(q-1)^{n-k}}{y_1 \cdots y_{k-1}} \sum_{i \geq 1} h_i[-X] g_i(\pi) \left[\frac{X}{q-1} \right].$$

On the other hand $g_i(\pi)$ were defined in such a way that

$$\chi_k(\pi)[(q-1)X] = \frac{(q-1)^{n-k}}{y_1 \cdots y_k} \sum_{i \geq 1} y_k^i g_i(\pi)[X + y_k].$$

Substituting $\frac{1}{q-1}X - y_k$ for X gives

$$(28) \quad \chi_k(\pi)[X - (q-1)y_k] = \frac{(q-1)^{n-k}}{y_1 \cdots y_k} \sum_{i \geq 1} y_k^i g_i(\pi) \left[\frac{X}{q-1} \right].$$

Comparing (27) and (28) we obtain

$$\chi_{k-1}(N\pi)[X] = \sum_{i \geq 0} -h_{i+1}[-X] \left(\chi_k(\pi)[X - (q-1)y_k] \Big|_{y_k^i} \right).$$

This can be seen to coincide with $d_- \chi(\pi)$, establishing the second case of (18). Thus the proof of Theorem 4.2 is complete.

4.5. Main recursion. We now show how to express all of $D_\alpha(q, t)$ using our operators:

Theorem 4.6. *If α is a composition of length l , we have*

$$D_\alpha(q, t) = d_-^l(N_\alpha).$$

where $N_\alpha \in V_l$ is defined by the recursion relations

$$(29) \quad N_\emptyset = 1, \quad N_{[1, \alpha]} = d_+ N_\alpha, \quad N_{a\alpha} = \frac{t^{a-1}}{q-1} [d_-, d_+] \sum_{\beta \models a-1} d_-^{l(\beta)-1} N_{\alpha\beta}.$$

Example 4.2. Using Theorem 4.6, we find that

$$\begin{aligned} N_{31} &= \frac{t^3}{(q-1)^2} (d_{-++-++} - d_{-++++-} - d_{+-+--+} + d_{+----+}) + \\ &\quad \frac{t^2}{q-1} (d_{-+-----} - d_{+-----}) = qt^3 y_1^2 - qt^2 y_1 e_1 \in V_2, \end{aligned}$$

where $d_{\epsilon_1 \dots \epsilon_n} = d_{\epsilon_1} \dots d_{\epsilon_n}(1)$. We may then check that

$$d_-^2 N_{31} = qt^3 B_3 B_1(1) + qt^2 B_2 B_1 B_1(1) = \nabla C_3 C_1(1).$$

Proof. For any $k > 0$ let $\mathbb{D}_k^0 \subset \mathbb{D}_k$ denote the subset of partial Dyck paths that begin with an East step. For $k = 0$ let $\mathbb{D}_0^0 = \{\emptyset\}$. Define functions $\chi^0 : \mathbb{D}_k^0 \rightarrow V_k$ by

$$\begin{aligned} \chi^0(\emptyset) &= 1, \quad \chi^0(EN^i \pi) = \frac{1}{q-1} [d_-, d_+] d_-^{i-1} \chi^0(\pi), \\ \chi^0(E\pi) &= d_+ \chi^0(\pi). \end{aligned}$$

Given a composition α of length l , let

$$\mathbb{D}_\alpha = \{\pi \in \mathbb{D} : \text{touch}'(\pi) = \alpha\}.$$

By the definition of touch' every element of \mathbb{D}_α is of the form $\pi = N^l \tilde{\pi}$ for a unique element $\tilde{\pi} \in \mathbb{D}_l^0$ so that by Corollary 4.3 we have

$$\chi(\pi, 0) = d_-^l \chi_l^0(\tilde{\pi}).$$

Let

$$N'_\alpha = \sum_{\pi \in \mathbb{D}_\alpha} t^{\text{bounce}(\pi)} \chi_l^0(\tilde{\pi}) \in V_l,$$

so that $D_\alpha(q, t) = d_-^l(N'_\alpha)$. It suffices to show that N'_α satisfies the relations (29), and so agrees with N_α .

For a composition α of length l and $0 \leq r \leq l$ we have a map $\gamma_{\alpha, r} : \mathbb{D}_\alpha \rightarrow \mathbb{D}$ as follows: $\gamma_{\alpha, r}(\pi) = N^{r+1} E N^{l-r} \tilde{\pi}$. Clearly $|\gamma_{\alpha, r}(\pi)| = |\pi| + 1$. From the definition of touch' we see the following relation:

$$\text{bounce}(\gamma_{\alpha, r}(\pi)) = \text{bounce}(\pi) + \sum_{i > r} \alpha_i.$$

Next we compute $\text{touch}'(\gamma_{\alpha, r}(\pi))$. For $0 \leq i \leq r$ we have

$$\text{bounce}(N^{i+2} E N^{r-i} E N^{l-r} \tilde{\pi}) = \text{bounce}(N^{i+1} E N^{l-i} \tilde{\pi}).$$

This implies

$$\text{touch}'(\gamma_{\alpha,r}(\pi)) = \left(1 + \sum_{i>r} \alpha_i, \alpha_1, \alpha_2, \dots, \alpha_r \right),$$

so in particular $\text{touch}'(\gamma_{\alpha,r}(\pi))$ depends only on α and r .

Since every non-empty Dyck path can be obtained as $\gamma_{\alpha,r}(\pi)$ in a unique way, we obtain for every composition α of length r :

$$\mathbb{D}_{a\alpha} = \bigsqcup_{\beta \models a-1} \gamma_{\alpha\beta,r}(\mathbb{D}_{\alpha\beta}).$$

It is not hard to see that this identity precisely translates to the relations (29) for N'_α . \square

5. OPERATOR RELATIONS

We have operators

$$(30) \quad e_k, d_\pm, T_i \subset V_* = V_0 \oplus V_1 \oplus \dots$$

where e_k is the projection onto V_k , the others are defined as above. It is natural to ask for a complete set of relations between them. They are formalized in the following algebra:

Definition 5.1. The Dyck path algebra $\mathbb{A} = \mathbb{A}_q$ (over R) is the path algebra of the quiver with vertex set $\mathbb{Z}_{\geq 0}$, arrows d_+ from i to $i+1$, arrows d_- from $i+1$ to i for $i \in \mathbb{Z}_{\geq 0}$, and loops T_1, T_2, \dots, T_{k-1} from k to k subject to the following relations:

$$\begin{aligned} (T_i - 1)(T_i + q) &= 0, \quad T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, \quad T_i T_j = T_j T_i \quad (|i - j| > 1), \\ T_i d_- &= d_- T_i, \quad d_+ T_i = T_{i+1} d_+, \quad T_1 d_+^2 = d_+^2, \quad d_-^2 T_{k-1} = d_-^2, \\ d_- (d_+ d_- - d_- d_+) T_{k-1} &= q(d_+ d_- - d_- d_+) d_- \quad (k \geq 2), \\ T_1 (d_+ d_- - d_- d_+) d_+ &= q d_+ (d_+ d_- - d_- d_+), \end{aligned}$$

where in each identity k denotes the index of the vertex where the respective paths begin. We have used the same letters T_i, d_\pm to label the i th loop at every node k to match with the previous notation. To distinguish between different nodes, we will use $T_i e_k$ where e_k is the idempotent associated with node k .

We will prove

Theorem 5.1. *The operators (30) define a representation of \mathbb{A} on V_* . Furthermore, we have an isomorphism of representations*

$$\varphi : \mathbb{A} e_0 \xrightarrow{\sim} V_*$$

which sends e_0 to $1 \in V_0$, and maps $e_k \mathbb{A} e_0$ isomorphically onto V_k .

The proof will occupy the rest of this section. We begin by establishing that we have defined a representation of the algebra.

Lemma 5.2. *The operators T_i and d_\pm satisfy the relations of Definition 5.1.*

Proof. The first line is just Proposition 4.1, and most follow from definition. The first one that does not is the commutation relation of d_+ with T_i . We have

$$\begin{aligned} d_+ T_i(F) &= T_1 \cdots T_k ((T_i F)[X + (q-1)y_{k+1}]) = \\ &= T_1 \cdots T_i T_{i+1} T_i \cdots T_k(F)[X + (q-1)y_{k+1}] = \\ &= T_1 \cdots T_{i+1} T_i T_{i+1} \cdots T_k(F)[X + (q-1)y_{k+1}] = T_{i+1} d_+(F) \end{aligned}$$

using the braiding relations.

For the next, we have

$$\begin{aligned} d_+^2 F &= T_1 T_2 \cdots T_{k+1} T_1 T_2 \cdots T_k(F[X + (q-1)y_{k+1} + (q-1)y_{k+2}]) = \\ &= T_2 T_3 \cdots T_{k+1} T_1 T_2 \cdots T_{k+1}(F[X + (q-1)y_{k+1} + (q-1)y_{k+2}]). \end{aligned}$$

The last T_{k+1} can be removed because its argument is symmetric in y_{k+1} and y_{k+2} , and we obtain $T_1^{-1} d_+^2 F$.

The next identity is more technical. The operator image of $T_{k-1} - 1$ consists of elements of the form $(qy_{k-1} - y_k)F$, where F is symmetric in y_{k-1} and y_k . Thus we need to check that d_-^2 vanishes on such elements. Let us evaluate d_-^2 on

$$(qy_{k-1} - y_k)y_{k-1}^a y_k^b F,$$

where F does not contain the variables y_{k-1} and y_k , and $a, b \in \mathbb{Z}_{\geq 0}$. We obtain

$$(qB_{a+2}B_{b+1} - B_{a+1}B_{b+2})F.$$

This expression is antisymmetric in a, b by Corollary 3.4 of [HMZ12], which implies our identity.

Next using the previous relations and Lemma 5.3 below write

$$\begin{aligned} d_-(d_+ d_- - d_- d_+) T_{k-1} &= (q-1) d_- T_1 T_2 \cdots T_{k-1} y_k T_{k-1} = \\ &= q(q-1) d_- T_1 T_2 \cdots T_{k-2} y_{k-1} = \\ (31) \quad &= q(q-1) T_1 T_2 \cdots T_{k-2} y_{k-1} d_- = q(d_+ d_- - d_- d_+) d_-. \end{aligned}$$

Similarly,

$$\begin{aligned} T_1(d_+ d_- - d_- d_+) d_+ &= (q-1) T_1 T_1 T_2 \cdots T_k y_{k+1} d_+ = \\ &= (q-1) q^k T_1 y_1 T_1^{-1} T_2^{-1} \cdots T_k^{-1} d_+ = (q-1) q^k T_1 y_1 T_1^{-1} d_+ T_1^{-1} \cdots T_{k-1}^{-1} = \\ (32) \quad &= (q-1) q^k d_+ y_1 T_1^{-1} \cdots T_{k-1}^{-1} = q d_+(d_+ d_- - d_- d_+). \end{aligned}$$

□

To establish the isomorphism, we first show that we can produce the operators of multiplication by y_i from \mathbb{A} .

Lemma 5.3. *For $F \in V_k$ we have*

$$(33) \quad (d_- d_+ - d_+ d_-)F = (q-1) T_1 T_2 \cdots T_{k-1} (-y_k F), \quad y_i = \frac{1}{q} T_i y_{i+1} T_i.$$

Proof. First, we endow V_k with the following twisted action of $\text{Sym}[X]$:

$$(F * G)[X] = F \left[X + (q-1) \sum_{i=1}^k y_i \right] G,$$

for $F \in \text{Sym}[X]$, and $G \in V_k$. It can be checked that the operators d_+ , d_- intertwine this action:

$$(34) \quad d_+(F * G) = F * d_+G, \quad d_-(F * G) = F * d_-G$$

We will not need this, but in fact, if $\pi_1 \in \mathbb{D}_k$, $\pi_2 \in \mathbb{D}$, and $\pi_1 \cdot \pi_2 \in \mathbb{D}_k$ is their concatenation, then we must also have that

$$\chi_k(w_1 \cdot w_2) = \chi(w_2) * \chi_k(w_1).$$

Since the operators on both sides commute with the twisted action of $\text{Sym}[X]$ introduced above, we may assume without loss of generality that F is a polynomial of y_1, y_2, \dots, y_k .

Write the left hand side of the first identity as

$$d_-T_1 \cdots T_{k-1}T_kF - T_1 \cdots T_{k-1}((d_-F)[X + (q-1)y_k]).$$

The operator d_- in the first summand involves only the variable y_{k+1} . Thus we can write the left hand side as

$$T_1 \cdots T_{k-1}(d_-T_kF - (d_-F)[X + (q-1)y_k]).$$

Hence it is enough to prove

$$d_-T_kF - (d_-F)[X + (q-1)y_k] = (1-q)y_kF.$$

It is clear that none of the operations involve the variables y_1, y_2, \dots, y_{k-1} . Thus we can assume $F = y_k^i$ for $i \in \mathbb{Z}_{\geq 0}$ without loss of generality. Direct computation gives

$$T_k(y_k^i) = y_{k+1}^i + (1-q) \sum_{j=1}^i y_k^j y_{k+1}^{i-j}.$$

Thus the left hand side equals

$$\begin{aligned} & -h_{i+1}[-X] - (1-q) \sum_{j=1}^i y_k^j h_{i-j+1}[-X] + h_{i+1}[-X - (q-1)y_k] = \\ & -(1-q) \sum_{j=1}^i y_k^j h_{i-j+1}[-X] + (1-q) \sum_{j=1}^{i+1} y_k^j h_{i-j+1}[-X] = (1-q)y_k^{i+1}. \end{aligned}$$

The second relation is easy. □

The operators of multiplication by y_i are characterized by these relations, and therefore come from elements of \mathbb{A} . We next establish the relations these operators satisfy within \mathbb{A} :

Lemma 5.4. For $k \in \mathbb{Z}_{>0}$ define elements $y_1, \dots, y_k \in e_k \mathbb{A} e_k$ by solving for $y_i F$ in the identities (33). Then the following identities hold in \mathbb{A} :

$$\begin{aligned} y_i T_j &= T_j y_i \quad \text{for } i \notin \{j, j+1\}, \\ y_i d_- &= d_- y_i, \quad d_+ y_i = T_1 T_2 \cdots T_i y_i (T_1 T_2 \cdots T_i)^{-1} d_+, \\ y_i y_j &= y_j y_i \quad \text{for any } i, j. \end{aligned}$$

Proof. Note that y_1 can be written as

$$y_1 = \frac{1}{q^{k-1}(q-1)}(d_+ d_- - d_- d_+) T_{k-1} \cdots T_1.$$

Our task becomes easier if we notice that it is enough to check the first identity for $i = 1$ and $i = k$, the second one for $i = k$, the third one for $i = 1$ and the last one for $i = 1, j = k$. The other cases can be deduced from these by applying the T -operators.

For $j > 1$ we have

$$\begin{aligned} y_1 T_j &= \frac{1}{q^{k-1}(q-1)}(d_+ d_- - d_- d_+) T_{k-1} \cdots T_1 T_j = \\ &= \frac{1}{q^{k-1}(q-1)}(d_+ d_- - d_- d_+) T_{j-1} T_{k-1} \cdots T_1 = T_j y_1. \end{aligned}$$

Similarly we verify that y_k commutes with T_j^{-1} hence with T_j for $j < k-1$.

Reversing the arguments in (31) and (32) we verify the second and the third identities.

Thus it is left to check that $y_k y_1 = y_1 y_k$. We assume $k \geq 2$. Write the left hand side as

$$y_k y_1 = \frac{1}{q-1} T_{k-1}^{-1} \cdots T_1^{-1} (d_+ d_- - d_- d_+) y_1 = \frac{1}{q-1} T_{k-1}^{-1} \cdots T_1^{-1} (T_1 y_1 T_1^{-1}) (d_+ d_- - d_- d_+)$$

using the already established commutation relations and that $k \geq 2$ to swap $T_1 y_1 T_1^{-1}$ and d_- . Performing the cancellation we obtain $y_1 y_k$. \square

The following lemma completes the proof of the theorem:

Lemma 5.5. The elements of the form

$$(35) \quad d_-^m y_1^{a_1} \cdots y_{k+m}^{a_{k+m}} d_+^{k+m} e_0$$

with $a_{k+1} \geq a_{k+2} \geq \cdots \geq a_{k+m}$ form a basis of $\mathbb{A} e_0$. Furthermore, the representation φ maps these elements to a basis of V_* .

Proof. We first show that elements of the form (35), with no condition on the a_i span \mathbb{A} . It suffices to check that the span of these elements is invariant under d_- , T_i and d_+ . This can be done by applying the following reduction rules that follow from the definition of \mathbb{A} and Lemma 5.4:

$$\begin{aligned} T_i d_- &\rightarrow d_- T_i, \quad T_j y_i \rightarrow y_i T_j \quad (i \notin \{j, j+1\}), \\ T_i y_i &\rightarrow y_{i+1} T_i + (1-q) y_i, \quad T_i y_{i+1} \rightarrow y_i T_i + (q-1) y_i, \\ T_i d_+^{k+m} e_0 &\rightarrow d_+^{k+m} e_0, \end{aligned}$$

$$d_+d_- \rightarrow d_-d_+ + (q-1)T_1T_2\cdots T_{k-1}y_k, \quad y_id_- \rightarrow d_-y_i.$$

The next step is to reduce the spanning set: We can use the following identity, which follows from $d_-^2T_{k-1} = d_-^2$:

$$d_-^m(1 - T_j)y_1^{a_1} \cdots y_{k+m}^{a_{k+m}}d_+^{k+m}e_0 = 0 \quad (k < j < k+m).$$

Note that T_j commutes with y_jy_{j+1} . Suppose $a_j < a_{j+1}$. Then we can rewrite the above identity as

$$0 = d_-^m y_1^{a_1} \cdots y_j^{a_j} y_{j+1}^{a_j} (1 - T_j) y_{j+1}^{a_{j+1}-a_j} y_{j+2}^{a_{j+2}} \cdots y_{k+m}^{a_{k+m}} d_+^{k+m} e_0.$$

Using $T_jy_{j+1} = y_j(T_j + (q-1))$, $T_jy_r = y_rT_j$ for $r > j+1$, and $T_jd_+^{k+m}e_0 = d_+^{k+m}e_0$ we can rewrite the identity as vanishing of a linear combination of terms of the form (35), and the lexicographically smallest term is precisely

$$d_-^m y_1^{a_1} \cdots y_{k+m}^{a_{k+m}} d_+^{k+m} e_0.$$

Thus we can always reduce terms of the form (35) which violate the condition $a_{k+1} \geq a_{k+2} \geq \cdots \geq a_{k+m}$ to a linear combination of lexicographically greater terms, showing that the subspace in the lemma at least spans $\mathbb{A}e_0$.

We now show that they map to a basis of V_* , which also establishes that they are independent, completing the proof. Consider the image of the elements of our spanning set

$$(36) \quad d_-^m y_1^{a_1} \cdots y_{k+m}^{a_{k+m}} d_+^{k+m} (1) = d_-^m (y_1^{a_1} \cdots y_{k+m}^{a_{k+m}}) (-1)^m y_1^{a_1} y_2^{a_2} \cdots y_k^{a_k} B_{a_{k+1}+1} B_{a_{k+2}+1} \cdots B_{a_{k+m}+1} (1).$$

Notice that $\lambda := (a_{k+1} + 1, a_{k+2} + 1, \dots, a_{k+m} + 1)$ is a partition, so

$$B_{a_{k+1}+1} B_{a_{k+2}+1} \cdots B_{a_{k+m}+1} (1)$$

is a multiple of the Hall-Littlewood polynomial $H_{\lambda'}[-X; 1/q, 0]$. These polynomials form a basis of the space of symmetric functions, thus the elements (36) form a basis of $\bigoplus_{k \in \mathbb{Z}_{\geq 0}} V_k$. □

6. CONJUGATE STRUCTURE

It is natural to ask if there is a way to extend ∇ to the spaces V_k , recovering the original operator at $k = 0$. What we have found is that it is better to extend the composition

$$(37) \quad \mathcal{N}(F) = \nabla \bar{\omega} F = \nabla \omega \bar{F}$$

where the conjugation simply makes the substitution $(q, t) = (q^{-1}, t^{-1})$, and $\omega(F) = F[-X]$ is the Weyl involution up to a sign, $\bar{\omega}$ denotes the composition of these. This is a very interesting operator, which in fact is an antilinear involution on $\text{Sym}[X]$ corresponding to dualizing vector bundles in the Haiman-Bridgeland-King-Reid picture, which identifies $\text{Sym}[X]$ with the equivariant K -theory of the Hilbert scheme of points in the complex plane [BKR01]. The key to our proof is to extend this operator to an antilinear involution on every V_k , suggesting that V_k should have some undiscovered geometric interpretation as well.

We will define the operator, which was discovered experimentally to have nice properties, by explicitly constructing the action of \mathbb{A} conjugated by the conjectural involution \mathcal{N} . Let $\mathbb{A}^* = \mathbb{A}_{q^{-1}}$, and label the corresponding generators by d_{\pm}^*, T_i^*, e_i^* . Denote by z_i the image of y_i under the isomorphism from \mathbb{A} to \mathbb{A}^* which sends generators to generators, and is antilinear with respect to $q \mapsto q^{-1}$.

Theorem 6.1. *There is an action of \mathbb{A}^* on V_* given by the assignment*

$$(38) \quad T_i^* = T_i^{-1}, \quad d_-^* = d_-, \quad e_i^* = e_i, \quad (d_+^* F)[X] = \gamma F[X + (q-1)y_{k+1}],$$

where $F \in V_k$ and γ is the operator which sends y_i to y_{i+1} for $i = 1, \dots, k$ and y_{k+1} to ty_1 . Furthermore, it satisfies the additional relations

$$(39) \quad z_{i+1}d_+ = d_+z_i, \quad y_{i+1}d_+^* = d_+^*y_i, \quad z_1d_+ = -y_1d_+^*tq^{k+1}, \quad d_+^*d_+^m(1) = d_+^{m+1}(1)$$

for any $m \geq 0$.

The statement is equivalent to validity of a certain set of relations satisfied by the operators. These will be verified in the following propositions.

First we list the obvious relations:

Proposition 6.2.

$$d_+^*T_i^{-1} = T_{i+1}^{-1}d_+^*, \quad T_1^{-1}d_+^{*2} = d_+^{*2}, \quad d_+^*y_i = y_{i+1}d_+^*.$$

Proof. Easy from the definition. □

To verify the rest of the relations, we use an approach similar to the one used in the proof of Lemma 5.3. Now we need not just one, but a family of twisted multiplications: For $F \in \text{Sym}[X]$, $G \in V_k$, $m = 0, 1, 2, \dots, k$ put

$$(F *_m G)[X] = F \left[X + (q-1) \left(\sum_{i=1}^m ty_i + \sum_{i=m+1}^k y_i \right) \right] G.$$

It is not hard to show that they satisfy

$$d_+^*(F *_m G) = F *_m d_+^*G, \quad d_-(F *_m G) = F *_m d_-G,$$

for $F \in \text{Sym}[X]$, and $G \in V_k$. Here the first identity holds for $0 \leq m \leq k$ and the second one for $0 \leq m < k$.

Let us first verify

Proposition 6.3.

$$(40) \quad d_-(d_+^*d_- - d_-d_+^*)T_{k-1}^{-1} = q^{-1}(d_+^*d_- - d_-d_+^*)d_- \quad (k \geq 2).$$

Proof. Rewrite it as

$$d_+^*d_-^2 - d_-d_+^*d_-(T_{k-1} + q) + qd_-^2d_+^* = 0.$$

Multiplying both sides by $q-1 = T_{k-1}-1+q-T_{k-1}$ produces an equivalent relation, which can be reduced to

$$(d_+^*d_-^2 - (q+1)d_-d_+^*d_- + qd_-^2d_+^*)(T_{k-1} + q) = 0.$$

Note that the image of $T_{k-1} + q$ consists of elements which are symmetric in y_{k-1}, y_k . Let

$$A = d_+^* d_-^2 - (q+1)d_- d_+^* d_- + qd_-^2 d_+^*.$$

It is enough to show that A vanishes on elements of V_k that are symmetric in y_{k-1}, y_k . We have (recall that $k \leq 2$)

$$A(F * G) = F *_1 A(G), \quad Ay_i = y_{i+1}A \quad (F \in \text{Sym}[X], G \in V_k, i < k-1),$$

Thus it is enough to verify vanishing of A on symmetric polynomials of y_{k-1}, y_k . We evaluate A on $y_{k-1}^a y_k^b$:

$$\begin{aligned} A(y_{k-1}^a y_k^b) &= (\Gamma_+(t(q-1)y_1)B_{a+1}B_{b+1} - (q+1)B_{a+1}\Gamma_+(t(q-1)y_1)B_{b+1} \\ &\quad + qB_{a+1}B_{b+1}\Gamma_+(t(q-1)y_1))1, \end{aligned}$$

where $\Gamma_+(Z)$ is the operator $F[X] \rightarrow F[X+Z]$. For any monomial u and integer i we have operator identities

$$\Gamma_+(u)B_i = (B_i - uB_{i-1})\Gamma_+(u), \quad B_i\Gamma_+(-u) = \Gamma_+(-u)(B_i - uB_{i-1}),$$

thus we have

$$\begin{aligned} \Gamma_+(t(q-1)y_1)B_{a+1}B_{b+1} &= \Gamma_+(-ty_1)(B_{a+1} - qty_1B_a)(B_{a+1} - qty_1B_a)\Gamma_+(qty_1), \\ B_{a+1}\Gamma_+(t(q-1)y_1)B_{b+1} &= \Gamma_+(-ty_1)(B_{a+1} - ty_1B_a)(B_{a+1} - qty_1B_a)\Gamma_+(qty_1), \\ B_{a+1}B_{b+1}\Gamma_+(t(q-1)y_1) &= \Gamma_+(-ty_1)(B_{a+1} - ty_1B_a)(B_{a+1} - ty_1B_a)\Gamma_+(qty_1). \end{aligned}$$

Performing the cancellations we arrive at

$$A(y_{k-1}^a y_k^b) = \Gamma_+(-ty_1)(ty_1(1-q)(B_aB_{b+1} - qB_{a+1}B_b))1.$$

This expression is antisymmetric in a, b by Corollary 3.4, [HMZ12]. Thus (40) is true. \square

Next we have to check that

Proposition 6.4.

$$T_1^{-1}(d_+^* d_- - d_- d_+^*)d_+^* = q^{-1}d_+^*(d_+^* d_- - d_- d_+^*).$$

Proof. Multiplying both sides by qT_1 and using the easier relations, rewrite it as

$$d_+^{*2}d_- - (T_1 + q)d_+^*d_-d_+^* + qd_-d_+^{*2} = 0.$$

Again, because of the commutation relations with the twisted multiplication by symmetric functions and y_i , it is enough to evaluate the left hand side on y_k^a for all $a \in \mathbb{Z}_{\geq 0}$. We obtain

$$-h_{a+1}[-X - t(q-1)(y_1 + y_2)] + (T_1 + q)h_{a+1}[-X - t(q-1)y_1] - qh_{a+1}[-X].$$

We use the identity $h_n[X+Y] = \sum_{i+j=n} h_i[X]h_j[Y]$ to write the left hand side as a linear combination of terms $h_{a+1-b}[-X]$ with $b > 0$. The coefficient in front of each term with $b > 0$ is

$$-h_b[t(1-q)(y_1 + y_2)] + (T_1 + q)h_b[t(1-q)y_1].$$

By a direct computation:

$$(T_1 + q)h_b[t(1-q)y_1] = (T_1 + q)(1-q)t^b y_1^b =$$

$$(1-q)t^b(y_1^b + (1-q)\sum_{i=1}^{b-1} y_1^i y_2^{b-i} + y_2^b) = h_b[t(1-q)(y_1 + y_2)],$$

and we are done. \square

At this point, we have established the fact that the operators given by (38) define an action of \mathbb{A}^* on V_* . Also we have established the second relation in (39). The last relation is obvious. The first and the third are verified below:

Proposition 6.5.

$$z_1 d_+ = -y_1 d_+^* t q^{k+1}, \quad z_{i+1} d_+ = d_+ z_i.$$

Proof. By definition (on V_k)

$$z_1 = \frac{q^{k-1}}{q^{-1} - 1} (d_+^* d_- - d_- d_+^*) T_{k-1}^{-1} \cdots T_1^{-1},$$

thus (again, on V_k)

$$z_1 d_+ = \frac{q^k}{q^{-1} - 1} (d_+^* d_- - d_- d_+^*) T_k^{-1} \cdots T_1^{-1} d_+.$$

From this expression the following two properties of $z_1 d_+$ are evident:

$$z_1 d_+ y_i = y_{i+1} z_1 d_+, \quad z_1 d_+ (F * G) = F *_1 z_1 d_+ (G)$$

for $F \in \text{Sym}[X]$, $G \in V_k$, $i = 1, \dots, k$. Similar properties are satisfied by $y_1 d_+^*$. Thus it is enough to verify the first identity on $1 \in V_k$. The right hand side is $-t q^{k+1} y_1$. The left hand side is

$$\frac{q^k}{q^{-1} - 1} (d_+^* - 1) d_-(1) = \frac{q^k}{q^{-1} - 1} (X + t(q-1)y_1 - X) = -t q^{k+1} y_1,$$

so the first identity holds.

It is enough to verify the second identity for $i = 1$ because the general case can be deduced follows from this one by applying the T -operators. For the identity $z_2 d_+ = d_+ z_1$, expressing z_1, z_2 in terms of d_-, d_+^* and the T -operators, we arrive at the following equivalent identity:

$$T_1^{-1} d_+ (d_+^* d_- - d_- d_+^*) = (d_+^* d_- - d_- d_+^*) d_+.$$

If we denote by A either of the two sides, we have

$$A(F * G) = F *_1 A(G), \quad A y_i = T_2 T_3 \cdots T_{i+1} y_{i+1} (T_2 T_3 \cdots T_{i+1})^{-1} A$$

for $F \in \text{Sym}[X]$, $G \in V_k$, $i = 1, \dots, k-1$. Thus it is enough to verify the identity on $y_k^a \in V_k$ ($a \in \mathbb{Z}_{\geq 0}$). Applying $T_k^{-1} T_{k-1}^{-1} \cdots T_2^{-1}$ to both sides, the identity to be verified is

$$T_k^{-1} T_{k-1}^{-1} \cdots T_1^{-1} d_+ (d_+^* d_- - d_- d_+^*) (y_k^a) = (d_+^* d_- - d_- d_+^*) T_{k-1}^{-1} \cdots T_1^{-1} d_+ (y_k^a).$$

The left hand side is evaluated to

$$-h_{a+1}[-X - t(q-1)y_1 - (q-1)y_{k+1}] + h_{a+1}[-X - (q-1)y_{k+1}].$$

The right hand side is evaluated to

$$(d_+^* d_- - d_- d_+^*) T_k(y_{k+1}^a) = F[X + t(q-1)y_1] - F[X]$$

with

$$\begin{aligned} F[X] &= -h_{a+1}[-X] - (1-q) \sum_{i=0}^{a-1} y_{k+1}^{a-i} h_{i+1}[-X] \\ &= -h_{a+1}[-X + (1-q)y_{k+1}] + (1-q)y_{k+1}^{a+1}, \end{aligned}$$

and the identity follows. \square

This completes our proof of Theorem 6.1.

We also have the following Proposition, which we will use to connect the conjugate action with N_α .

Proposition 6.6. *For a composition α of length k let*

$$y_\alpha = y_1^{\alpha_1-1} \cdots y_k^{\alpha_k-1} \in V_k.$$

Then the following recursions hold:

$$y_{1\alpha} = d_+^* y_\alpha, \quad y_{a\alpha} = \frac{t^{1-a}}{q-1} (d_+^* d_- - d_- d_+^*) \sum_{\beta \models a-1} q^{1-l(\beta)} d_-^{l(\beta)-1} (y_{\alpha\beta}) \quad (a > 1).$$

Proof. The first identity easily follows from the explicit formula for d_+^* . For $i = 1, 2, \dots, k-1$ we have

$$(d_- d_+^* - d_+^* d_-) y_i = y_{i+1} (d_- d_+^* - d_+^* d_-).$$

Therefore it is enough to verify the following identity for any $a \in \mathbb{Z}_{\geq 1}$:

$$(41) \quad (q-1)t^a y_1^a = (d_+^* d_- - d_- d_+^*) \sum_{\beta \models a} q^{1-l(\beta)} d_-^{l(\beta)-1} (y_k^{\beta_1-1} \cdots y_{k+l(\beta)-1}^{\beta_{l(\beta)}-1}) \in V_k.$$

We group the terms on the right hand side by $b = \beta_1 - 1$ and the sum becomes

$$\begin{aligned} & \sum_{b=0}^{a-1} y_k^b \sum_{\beta \models a-b-1} q^{-l(\beta)} d_-^{l(\beta)} \left(y_{k+1}^{\beta_1-1} \cdots y_{k+l(\beta)}^{\beta_{l(\beta)}-1} \right) \\ &= \sum_{b=0}^{a-1} y_k^b \sum_{\beta \models a-b-1} q^{-l(\beta)} (-1)^{l(\beta)} B_{\beta_1} \cdots B_{\beta_{l(\beta)}}(1) = \sum_{b=0}^{a-1} y_k^b h_{a-b-1}[q^{-1}X]. \end{aligned}$$

We have used the identity

$$(42) \quad h_n[q^{-1}X] = \sum_{\alpha \models n} q^{-l(\alpha)} (-1)^{l(\alpha)} B_\alpha(1),$$

which can be obtained by applying $\bar{\omega}$ to Proposition 5.2 of [HMZ12]:

$$h_n[-X] = \sum_{\alpha \models n} C_\alpha(1).$$

Thus the right hand side of (41) is evaluated to the following expression:

$$\begin{aligned}
& (d_+^* d_- - d_- d_+^*) \sum_{b=0}^{a-1} y_k^b q^{-(a-b-1)} h_{a-b-1}[X] \\
&= - \sum_{b=0}^{a-1} (\Gamma_+(t(q-1)y_1) B_{b+1} - B_{b+1} \Gamma_+(t(q-1)y_1)) h_{a-b-1}[q^{-1}X]. \\
&= - \sum_{b=0}^{a-1} \Gamma_+(-ty_1) ((B_{b+1} - qty_1 B_b) - (B_{b+1} - ty_1 B_b)) (h_{a-b-1}[q^{-1}X + ty_1]) \\
&= (q-1)ty_1 \Gamma_+(-ty_1) \sum_{b=0}^{a-1} B_b(h_{a-b-1}[q^{-1}X + ty_1]).
\end{aligned}$$

Thus we need to prove

$$\sum_{b=0}^{a-1} B_b(h_{a-b-1}[q^{-1}X + ty_1]) = t^{a-1} y_1^{a-1}.$$

Then the left hand side as a polynomial in y_1 indeed has the right coefficient of y_1^{a-1} . The coefficient of y_1^i for $i < a-1$ is

$$t^i \sum_{b=0}^{a-1-i} B_b(h_{a-b-1-i}[q^{-1}X]).$$

So it is enough to show:

$$\sum_{b=0}^m B_b(h_{m-b}[q^{-1}X]) = 0 \quad (m \in \mathbb{Z}_{>0}).$$

Using (42) again we see that the left hand side equals

$$B_0(h_m[q^{-1}X]) - qh_m[q^{-1}X] = (B_0 - q)(-q^{-1}C_m(1)) = 0$$

because $B_0 C_m = q C_m B_0$ by Proposition 3.5 of [HMZ12] and $B_0(1) = 1$. \square

7. THE INVOLUTION

Definition 7.1. Consider \mathbb{A} and \mathbb{A}^* as algebras over $\mathbb{Q}(q, t)$, and let $\tilde{\mathbb{A}} = \tilde{\mathbb{A}}_{q,t}$ be the quotient of the free product of \mathbb{A} and \mathbb{A}^* by the relations

$$\begin{aligned}
d_-^* &= d_-, \quad T_i^* = T_i^{-1}, \quad e_i^* = e_i, \\
z_{i+1} d_+ &= d_+ z_i, \quad y_{i+1} d_+^* = d_+^* y_i, \quad z_1 d_+ = -y_1 d_+^* t q^{k+1},
\end{aligned}$$

We now prove

Theorem 7.1. *The operations T_i , d_- , d_+ , d_+^* , e_i define an action of $\tilde{\mathbb{A}}$ on V_* . Furthermore, the kernel of the natural map $\tilde{\mathbb{A}}e_0 \rightarrow V_*$ that sends fe_0 to $f(1)$ is given by Ie_0 where $I \subset \tilde{\mathbb{A}}$ is the ideal generated by*

$$(43) \quad I = \langle d_+^* d_+^m - d_+^{m+1} \mid m \geq 0 \rangle.$$

In particular, we have an isomorphism $V_ \cong \tilde{\mathbb{A}}e_0/Ie_0$.*

Proof. Theorem 6.1 shows that we have a map of modules $\tilde{\mathbb{A}}e_0 \rightarrow V_*$, that restricts to the isomorphism of Theorem 5.1 on the subspace $\mathbb{A}e_0$, so in particular is surjective. Furthermore, the last relation of (39) shows that it descends to a map $\tilde{\mathbb{A}}e_0/Ie_0 \mapsto V_*$, which must still be surjective. We have the following commutative diagram:

$$\begin{array}{ccc} \tilde{\mathbb{A}}e_0/Ie_0 & \xrightarrow{\quad} & V_* \\ \uparrow & \nearrow \sim & \\ \mathbb{A}e_0 & & \end{array}$$

Thus we have an inclusion $\mathbb{A}e_0 \subset \tilde{\mathbb{A}}e_0/Ie_0$ and it remains to show that the image of $\mathbb{A}e_0$ in $\tilde{\mathbb{A}}e_0/Ie_0$ is the entire space. We do so by induction: notice that both $\mathbb{A}e_0$ and $\tilde{\mathbb{A}}e_0/Ie_0$ have a grading by the total degree in d_+ , d_+^* and d_- , as all the relations are homogeneous. For instance, y_i and z_i have degree 2, and T_i has degree 0 for all i . Denote the space of elements of degree m in $\mathbb{A}e_0$, $\tilde{\mathbb{A}}e_0/Ie_0$ by $V^{(m)}$, $W^{(m)}$ respectively. We need to prove $V^{(m)} = W^{(m)}$. The base cases $m = 0$, $m = 1$ are clear.

For the induction step, suppose $m > 0$, $V^{(i)} = W^{(i)}$ for $i \leq m$ and let $F \in V^{(m)}$. It is enough to show that $d_+^*F \in V^{(m+1)}$. By Lemma 5.5, we can assume that F is in the canonical form (35). We therefore must check three cases: $F = d_+^m(1)$ for $1 \in V_0$, $F = y_i G$ for $G \in V^{(m-2)}$, and $F = d_-(G)$ for $G \in V^{(m-1)}$. In the first case we have $d_+^*F = d_+^{m+1}1$. In the second case we have $d_+^*(F) = y_{i+1}d_+^*(G)$.

In the third case, we have

$$d_+^*F = d_+^*d_-(G) = d_-d_+^*G + (q^{-1} - 1)T_1^{-1} \cdots T_{k-1}^{-1}z_k G.$$

Now we use expansion of G in terms of the generators T_i , d_+ and d_- . Because of the commutation relations between T_i and z_j it is enough to consider two cases: $G = d_+G'$ and $G = d_-G'$ for $G' \in V^{(m-2)}$. In the first case we have $z_k G = d_+z_{k-1}G'$ if $k > 1$ and $z_1 G = -y_1 d_+^* t G'$ if $k = 1$. In the second case we have $z_k G = d_-z_k G'$. In all cases the claim is reduced to the induction hypothesis. \square

Now by looking at the defining relations of $\tilde{\mathbb{A}}$, we make the remarkable observation that there exists an involution ι of $\tilde{\mathbb{A}}$ that permutes \mathbb{A} and \mathbb{A}^* and is antilinear with respect to the conjugation $(q, t) \mapsto (q^{-1}, t^{-1})$ on the ground field $\mathbb{Q}(q, t)$! Furthermore, this involution preserves the ideal I , and therefore induces an involution on V_* via the isomorphism of Theorem 7.1.

Theorem 7.2. *There exists a unique antilinear degree-preserving automorphism $\mathcal{N} : V_* \rightarrow V_*$ satisfying*

$$\mathcal{N}(1) = 1, \quad \mathcal{N}T_i = T_i^{-1}\mathcal{N}, \quad \mathcal{N}d_- = d_-\mathcal{N}, \quad \mathcal{N}d_+ = d_+^*\mathcal{N}, \quad \mathcal{N}y_i = z_i\mathcal{N}.$$

Moreover, we have

- (i) \mathcal{N} is an involution, i.e. $\mathcal{N}^2 = \text{Id}$.
- (ii) For any composition α we have

$$\mathcal{N}(y_\alpha) = q^{\sum(\alpha_i - 1)} N_\alpha.$$

(iii) On $V_0 = \text{Sym}[X]$, we have $\mathcal{N} = \nabla \bar{\omega}$, where $\bar{\omega}$ is the involution sending q , t , X to q^{-1} , t^{-1} , $-X$ resp. (see (37)).

Proof. The automorphism is induced from the involution of $\tilde{\mathbb{A}}$, from which part (i) follows immediately. Part (ii) follows from applying \mathcal{N} to the relations of Proposition 6.6.

Finally, let $D_1, D_1^* : V_0 \rightarrow V_0$ be the operators

$$(D_1 F)[X] = F[X + (1 - q)(1 - t)u^{-1}] \text{Exp}[-uX]|_{u^1},$$

$$(D_1^* F)[X] = F[X - (1 - q^{-1})(1 - t^{-1})u^{-1}] \text{Exp}[uX]|_{u^1},$$

and let $\underline{e}_1 : V_0 \rightarrow V_0$ be the operator of multiplication by $e_1[X] = X$. It is easy to verify that

$$D_1 = -d_- d_+^*, \quad \underline{e}_1 = d_- d_+, \quad \bar{\omega} D_1 = D_1^* \bar{\omega}.$$

Thus it follows that

$$\mathcal{N} D_1 = -\underline{e}_1 \mathcal{N}, \quad \mathcal{N} \underline{e}_1 = -D_1 \mathcal{N}.$$

Let $\nabla' = \mathcal{N} \bar{\omega}$ on V_0 . Then

$$\nabla'(1) = 1, \quad \nabla' \underline{e}_1 = D_1 \nabla', \quad \nabla' D_1^* = -\underline{e}_1 \nabla'.$$

It was shown in [GHT99] that ∇ satisfies the same commutation relations, and that one can obtain all symmetric functions starting from 1 and successively applying \underline{e}_1 and D_1^* . Thus $\nabla = \nabla'$, proving part (iii). \square

The compositional shuffle conjecture now follows easily:

Theorem 7.3. *For a composition α of length k , we have*

$$\nabla C_{\alpha_1} \cdots C_{\alpha_k}(1) = D_\alpha(X; q, t).$$

Proof. Using Theorems 4.6 and 7.2, we have

$$\begin{aligned} D_\alpha(q, t) &= d_-^k(N_\alpha) = d_-^k(\mathcal{N}(q^{|\alpha|-k} y_\alpha)) = \mathcal{N}(q^{|\alpha|-k} d_-^k(y_\alpha)) = \\ &= \mathcal{N}(q^{|\alpha|-k} (-1)^k B_\alpha(1)) = \mathcal{N} \bar{\omega} C_\alpha(1) = \nabla C_\alpha(1). \end{aligned}$$

\square

REFERENCES

- [BGHT99] F. Bergeron, A. M. Garsia, M. Haiman, and G. Tesler. “Identities and Positivity Conjectures for Some Remarkable Operators in the Theory of Symmetric Functions”. *Methods and Applications of Analysis* 3 (6 1999), pp. 363–420.
- [BGLX14] F. Bergeron, A. Garsia, E. Leven, and G. Xin. “Compositional (km, kn) -Shuffle Conjectures” (2014). arXiv:1404.4616.
- [BKR01] T. Bridgeland, A. King, and M. Reid. “The McKay correspondence as an equivalence of derived categories”. *J. Amer. Math. Soc.* 14 (2001), pp. 535–554.
- [EHKK03] E. Egge, J. Haglund, K. Killpatrick, and D. Kremer. “A Schröder generalization of Haglund’s statistic on Catalan paths”. *Electron. J. Combin.* 10 (2003).

- [GH02] A. Garsia and J. Haglund. “A proof of the q, t -Catalan positivity conjecture”. *Discrete Math* (2002), pp. 677–717.
- [GHT99] A. M. Garsia, M. Haiman, and G. Tesler. “Explicit plethystic formulas for Macdonald q, t -Kostka coefficients”. *Sém. Lothar. Combin.* 42 (1999). The Andrews Festschrift (Maratea, 1998), Art. B42m, 45.
- [GM96] A. Garsia and H. M. “A Remarkable q, t -Catalan Sequence and q -Lagrange Inversion”. *Journal of Algebraic Combinatorics* (1996), pp. 191–244.
- [GN15] E. Gorsky and A. Negut. “Refined knot invariants and Hilbert schemes”. *Journal de Mathématiques Pures et Appliquées* 104.3 (2015), pp. 403–435.
- [GORS14] E. Gorsky, A. Oblomkov, J. Rasmussen, and V. Shende. “Torus knots and the rational DAHA”. *Duke Mathematical Journal* 14 (163 2014), pp. 2709–2794.
- [Hag03] J. Haglund. “Conjectured statistics for the q, t -Catalan numbers”. *Adv. Math.* 175 (2003).
- [Hag04] J. Haglund. “A proof of the q, t -Schröder conjecture”. *Internat. Math. Res. Notices* 11 (2004), pp. 525–560.
- [Hag08] J. Haglund. *The q, t -Catalan numbers and the space of diagonal harmonics*. Vol. 41. University Lecture Series. With an appendix on the combinatorics of Macdonald polynomials. American Mathematical Society, Providence, RI, 2008, pp. viii+167.
- [Hai01] M. Haiman. “Hilbert schemes, polygraphs, and the Macdonald positivity conjecture”. *J. Amer. Math. Soc.* 14 (2001), pp. 941–1006.
- [Hai02] M. Haiman. “Vanishing theorems and character formulas for the Hilbert scheme of points in the plane”. *Invent. Math.* 149 (2002), pp. 371–407.
- [HHL05] J. Haglund, M. Haiman, and N. Loehr. “A combinatorial formula for Macdonald polynomials”. *Journal of the American Mathematical Society* 18.3 (2005), pp. 735–761.
- [HHLRU05] J. Haglund, M. Haiman, N. Loehr, J. Remmel, and A. Ulyanov. “A combinatorial formula for the character of the diagonal coinvariants”. *Duke J. Math.* (126 2005), pp. 195–232.
- [Hic12] A. Hicks. “Two Parking Function Bijections: A Sharpening of the q, t -Catalan and Schröder Theorems”. *Int. Math. Res. Notices* 2012 (2012), pp. 3064–3088.
- [Hik14] T. Hikita. “Affine Springer fibers of type A and combinatorics of diagonal coinvariants”. *Adv. Math.* 263 (2014), pp. 88–122.
- [HMZ12] J. Haglund, J. Morse, and M. Zabrocki. “A compositional shuffle conjecture specifying touch points of the Dyck path”. *Canad. J. Math.* 64.4 (2012), pp. 822–844. DOI: 10.4153/CJM-2011-078-4.
- [Mac95] I. G. MacDonald. *Symmetric functions and Hall polynomials*. Oxford Mathematical Monographs. Oxford: Clarendon Press, 1995.
- [Neg13] A. Negut. “The shuffle algebra revisited”. *International research notices* (2013).

- [SV11] O. Schiffmann and E. Vasserot. “The elliptical Hall algebra, Cherednik Hecke algebras, and Macdonald polynomials”. *Compos. Math.* 147.1 (2011), pp. 188–234.
- [SV13] O. Schiffmann and E. Vasserot. “The elliptical Hall algebra and the K -theory of the Hilbert scheme of \mathbb{A}^2 ”. *Duke Math. J.* 162.2 (2013), pp. 279–366.